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Exam in Discrete Mathematics, 7,5 hp.
Tuesday 11th of June 2019, Time 08.00-13.00.

To obtain maximal points a complete solution, presented in such a way that calculations and reasoning are easy to follow, is demanded. If nothing else is given you can assume that all graphs are undirected, loopfree and not multigraphs.
Aid: Sheet with formulas.

1. Consider passwords with three symbols. Allowed symbols are the 26 letters in the English alphabet and the ten digits 0 to 9. We put the following restrictions on the passwords:

   i) There must be both letters and digits.
   ii) There must be both upper- and lower-case letters.

   So K8k and 4zL are examples of allowed passwords. How many such passwords can be constructed? (3p)

2. Ten short questions so the motivations can also be short. One point on each.

   a) Give the number of non-negative integer solutions to the equation $x_1+x_2+x_3+x_4 = 14$.

   b) Give a Hasse diagram for a *totally ordered set* (TOS) with five elements. For every pair a and b in a TOS we have aRb or bRa.

   c) Show in a Venn diagram the set $A \cap \overline{B}$. A and B are two different non-empty sets with an intersection. $\overline{B}$ denotes the complement to set B.

   d) Give the generating function for the sequence $1, 5, 10, 10, 5, 1, 0, 0, 0, 0, 0 \cdots$.

   e) In which complete graphs, K_n $(n \geq 3)$, can we find an Euler circuit?

   f) In which complete bipartite graphs can we find a Hamilton cycle?

   g) How many proper colorings can be made of the vertices in C_3 if 5 colors are available?

   h) How many bit strings of length seven are there?

   i) Draw a tree with 3 internal vertices and 2 leaves.

   j) How many functions are there from the set of bit strings of length 3 to the set of bit strings of length 1? (10p)

3. Use the inclusion-exclusion (IE) method to calculate the number of integers between 1 and 60 that are *coprime* with 60. The primes 2, 3 and 5 are the prime factors in 60 since $60 = 2^2 \cdot 3 \cdot 5$. The integers, i, that are coprime with 60 are *not* divisible with 2, 3 and 5 so $GCD(i, 60) = 1$ for them. GCD stands for greatest common divisor. You can count them directly but here you have to use the IE-method. (5p)

4. Let $X = \{1, 2, 3, 4\}$ and $A = X \times X$. Define a relation $\mathcal{R}$ on A by $(a, b)\mathcal{R}(c, d)$ if and only if $ab = cd$. This relation, $\mathcal{R}$, is an equivalence relation. Determine the equivalence classes. (3p)

5. The *girth* (omkrets in Swedish) of a graph G is the number of edges of the shortest cycle it is possible to find in G and it is denoted $g(G)$. Let, from now on, the graphs G have $e = v + 1$ For such graphs with $v \geq 4$ it is possible to show that

$$g(G) \leq \left\lfloor \frac{2(v+1)}{3} \right\rfloor .$$

The floor function $\lfloor x \rfloor$ gives the largest integer $\leq x$.

a) Explain why in this type of graphs there must be at least two cycles. Draw a case where the cycles are edge disjoint for $v = 7$ and $e = 8$. (2p)

b) If the two cycles are edge disjoint what is the maximum girth in a graph with $e = v+1$? (1p)

c) Draw a graph with maximal girth for $v = 7$ and $e = 8$. (2p)

6. a) What is the maximal number of edges in a planar graph with v vertices when the regions have degree 3 or more? Use for example $2e = \Sigma \deg(R_i)$ and Euler's formula. (2p)

b) Let G be a graph with 11 vertices and let $\bar{G}$ be its complement graph. $\bar{G}$ has the same 11 vertices but the edges from K_{11} that are *not* edges in G. Use the result in (a) above to show that not both G and $\bar{G}$ can be planar. (2p)

Good Luck!