Discrete Mathematics,
chapter 1, Logic & Proofs

Predicate logic

Propositional logic

"Volvo is an Italian car"

"The tomato contains water"

Sections 1.6 and 1.7 about proofs are readable! For the rest we take a quick overview.

1.1) p, q propositions

$\neg$ Negation

$\wedge$, $\vee$ connectives

Truth Table

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Implications $p \rightarrow q$, $q \rightarrow p$

$p \longleftrightarrow q$, if and only if

Ex) Logical puzzle from Smullyan.
Knights tell the truth.
Knaves always lie.
A: "B is a knight"
B: "The two of us are opposite types"
Can you tell the type of A and B.

1.2) Tautology - a compound proposition that is always true. $p \vee \neg p$
Contradiction - a compound proposition that is always false.
$p \wedge \neg p$

De Morgan's laws:

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$
$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

Check the truth tables.

1.3) $P(x)$: $x < 7$

x - variable
$x < 7$ predicate
$P(x)$ - propositional function.

$P(5)$ true
$P(9)$ false

Quantifiers: $\forall x\, P(x)$
$\exists x\, P(x)$

Ex) The domain is $\mathbb{R}$

$\exists x \; (x^3 = -1)$ True!

$\exists x \; (x^4 < x^2)$ True!

$\forall x \; ((-x)^2 = x^2)$

$\forall x \; (2x > x)$

Nested quantifiers (1.4)

$\forall x \, \exists y \; (x + y = 0)$

Ex) Domain $\mathbb{Z}$

$\forall x \, \exists y \; (x = \frac{1}{y})$ False!

Counterexample $x = 7$. $\frac{1}{7} \notin \mathbb{Z}$

Rules of <u>Inference</u> (1.5)

<u>Proofs</u> in mathematics are <u>valid</u> <u>arguments</u> that establish the truth of mathematical statements.

Valid arguments (see Table 1)

$$\begin{array}{l} \neg q \\ p \to q \\ \hline \therefore \neg p \end{array} \qquad [\neg q \wedge (p \to q)] \to \neg p$$

1.6 Introduction to (informal) proofs

$\boxed{\text{Axioms (postulates), definitions.}}$

$\boxed{\text{Theorems, propositions, lemma, Corollary,}}$

Think of Euclide's Elements.
aleph0.clark.edu/~djoyce/java/elements/elements.html

BUT! "such presentations do not convey the discovery process in mathematics" (page 97)

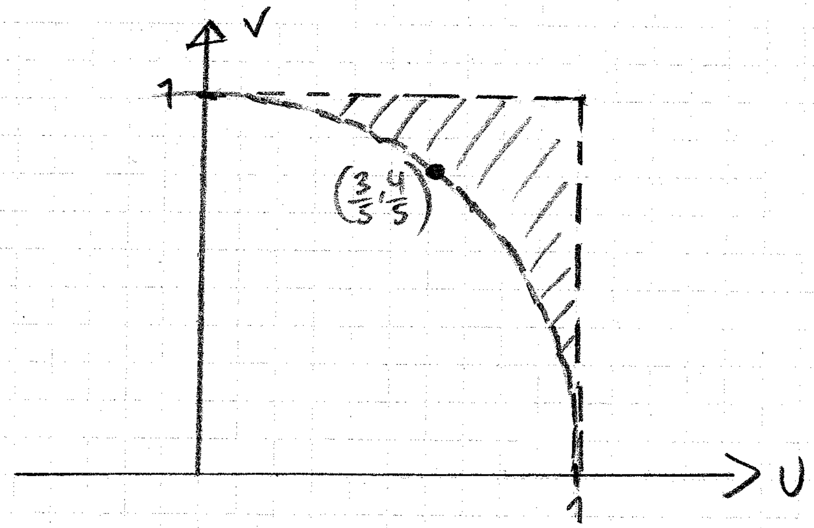
* Direct Proof $p \to q$
* Proof by contraposition, $\neg q \to \neg p$
* Proof by contradiction
* Proof by cases
* Existence Proofs
* Uniqueness Proofs

Open Problems:

* Riemann hypothesis (1859)
* Goldbach's conjecture. $4 = 2+2$, $6 = 3+3$, $8 = 5+3$, $10 = 5+5$, $12 = 7+5$
* 3x+1 conjecture. Ex) $11 \to 34 \to 17 \to 52 \to 26 \to 13 \to 40 \to 20 \to 10 \to 5 \to 16 \to 8 \to 4 \to 2 \to 1 \to 4 \to 2 \to 1 \to \ldots$

1.7.30) $3^2+4^2=5^2$
$5^2+12^2=13^2$ Are there more integer solutions to $x^2+y^2=z^2$?

If so $u^2+v^2=1$, $u=x/z$ and $v=y/z$ are rational numbers.



Yes! Put $x=m^2-n^2$, $y=2mn$ and $z=m^2+n^2$, m and n are integers
Check that $x^2+y^2=z^2$

Are there integer solutions to

$$x^3+y^3=z^3$$ or even

higher exponents 4, 5, ... ?

No! (Wiles 1994)

The curves $u^n+v^n=1$ $n=3,4,5\ldots$ will lie in the shaded region. Will never hit a rational point!