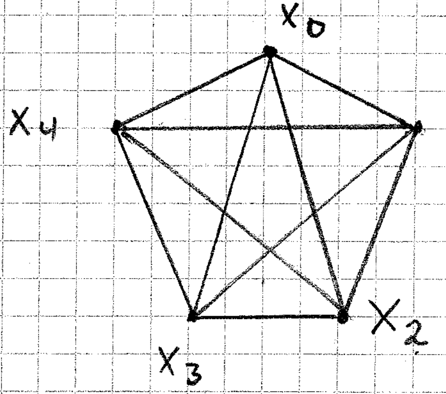
Graphs & Trees, part II.

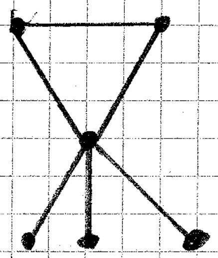
K_5



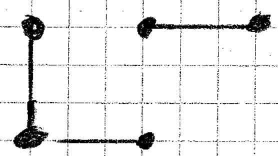
x_0, x_1, x_3 a <u>path</u> of length 2.

$x_2, x_4, x_1, x_2, x_4, x_3$ a, not simple, path of length 5.
x_4, x_1, x_0, x_4 is a <u>circuit</u>, or cycle, of length 3.

G_1



G_2

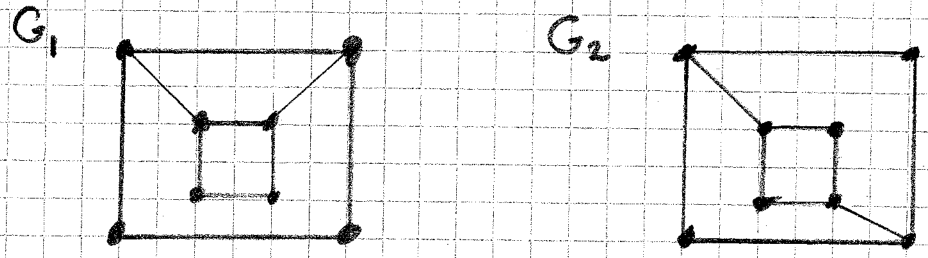


<u>connected</u>, that is, there is a path between any pair of vertices.

not connected but G_2 has 2 connected components.

WWW-graph has a connected component that includes $\approx 90\%$ of the vertices. (1999)

Ex) 9.4.18 G_1 and G_2 isomorphic?

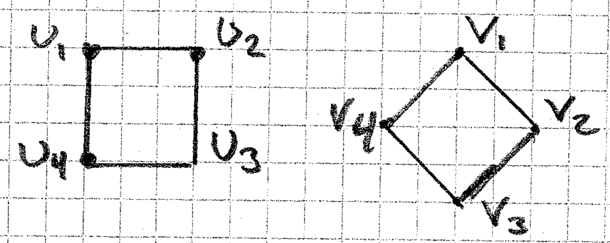


$|V_1| = 8$ $\quad$ $|V_2| = 8$

$|E_1| = 10$ $\quad$ $|E_2| = 10$

# vertices with degree		
2	4	4
3	4	4

So far so good but they don't look isomorphic! Let's have a look at circuits also. They are also isomorphic <u>invariants</u>.

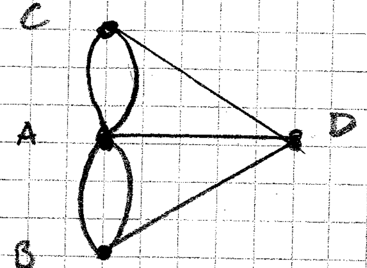


# circuits of length	in G_1	in G_2
4	3	2
5	0	0
6	2	4

$\Rightarrow$ G_1 and G_2 are not isomorphic.

9.5 Euler and Hamilton paths.

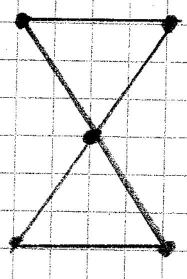
G



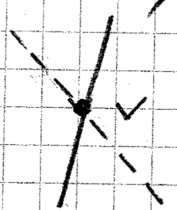
Is there a simple circuit passing over all edges of G?

Such a circuit is called an <u>Euler circuit</u>. (EC)

Compare with H

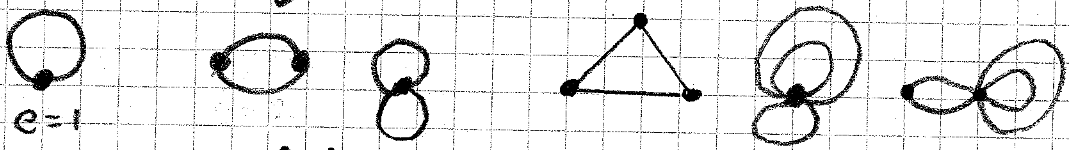


Necessary condition for EC is $\deg(v) = 2k$, $k \in \mathbb{Z}^+$, $\forall v \in V$.



2 edges for each visit at v.

To show that this is also a sufficient condition is harder. Use induction over the number of edges. See appendix.

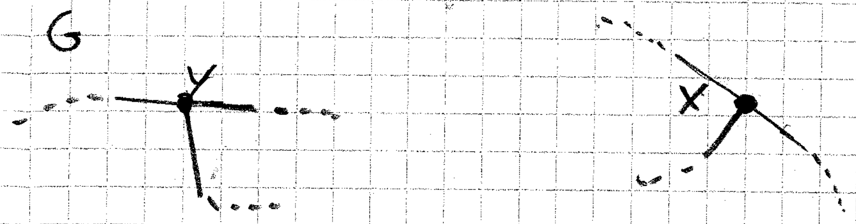


$e=1$ $\quad$ $e=2$ $\quad$ $e=3$

2 $\quad$ $4 = 2+2$ $\quad$ $6 = 2+2+2 = 4+2$

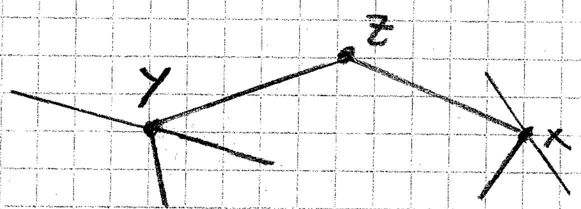
A connected multigraph G has an Euler path, but no Euler circuit, iff G has exactly two vertices of odd degree.

G



$\Rightarrow$ $\deg(x) = 1+2+2+\ldots$
$\deg(y) = 2+2+\ldots+1$
$\deg(v) = 2+2+2$ $v \neq x$ and y

$\Leftarrow$ Construct from G a new graph G^* with one more vertex z and two more edges, $\{y, z\}$ and $\{z, x\}$.

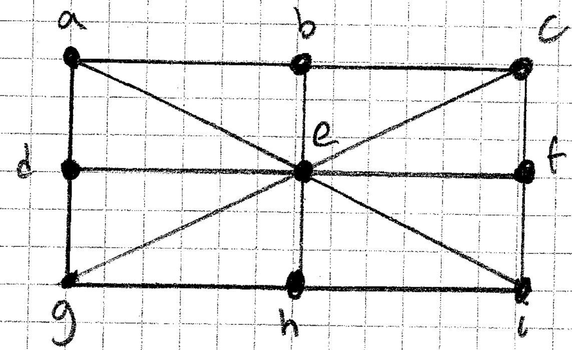


From previous page we know that G^* has an EC. So there is an EP from y to x.

Hamilton paths and Hamilton circuits (HC).

A simple circuit that passes through every vertex exactly once is called an HC.

Ex) 9.5.36



Find a route for the travelling salesman!

No useful necessary and sufficient condition for HC is known. A sufficient condition is:

Dirac's theorem: G is simple and $|V| = n$, $n \geq 3$. If $\deg(v) \geq \frac{n}{2}$ for all $v \in V$ then G has an HC.

In the graph above $n = 9$ but $\deg(v) = 3$ for all vertice except e.

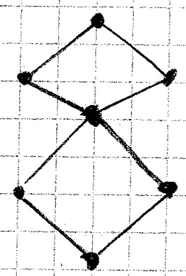
The (worst case) time complexity is exponential for finding HC. It is an NP-complete problem.

Appendix. Final part of induction proof of:

$\deg(v)$ even, $\forall v \in V \Rightarrow$
There is an EC in G.

Proof: Since the degrees are even it follows that G has a cycle as subgraph.

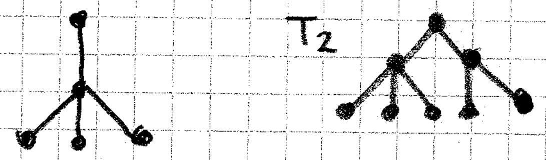
Here is an illustration



We can understand this in the following way:

A <u>tree</u> has by definition no cycles

T_1 T_2



$|V|=5$ $|V|=8$

$|E|=4$ $|E|=7$

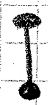
Moreover for trees we have

$$|V| = |E| + 1 \qquad (*)$$

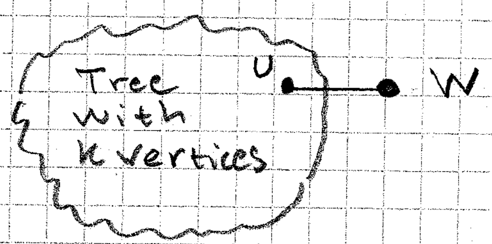
A proof of $(*)$ is given in 10.1

Induction over vertices this time.

$|V|=1$ $|V|=2$

$|E|=0$  $|E|=1$ 

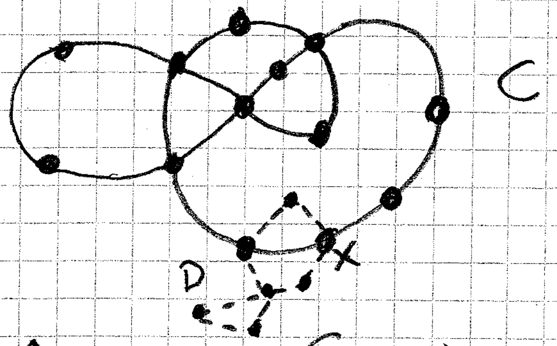
Assume it holds for $|V|=k$, Consider a tree with $k+1$ vertices ....



But if $\deg(v)$ even for all vertices then

$$2|E| = \sum_{v\in V} \deg(v) \geq 2|V|$$

so in these graphs $|V| \leq |E|$. They are not trees! There are cycles,



Let now C be a circuit in G with maximal number of edges.

Assume C is not an EC. Then there is an $x \in C$ which belongs to a component D but every vertex in D has even degree so by the induction assumption there is an EC in D. Join it to C! C had not maximal number of edges! If there are even more componets join them to the circuit. At last it becomes an EC!

From
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Modern Graph Theory