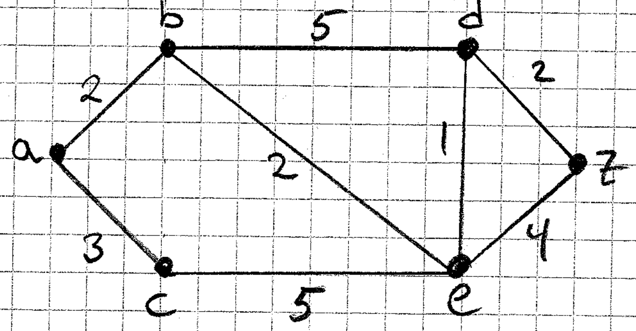
Graphs and Trees, part III.

9.6 Shortest path Problems

Ex)



A weighted graph

The weights can be distance, time, cost ....

Without weights there are 2 paths of length 3 from a to z.

Which path from a to z minimize the sum of the weights?

Dijkstras algorithm

Step	path	Σ	vertices included
1	a, b	2	a, b
2	a, c	2	a, b, c
3	a, b, e	4	a, bc, e
4	a, b, e, d	5	a, b, c, de
5	a, b, e, d, z	7	all

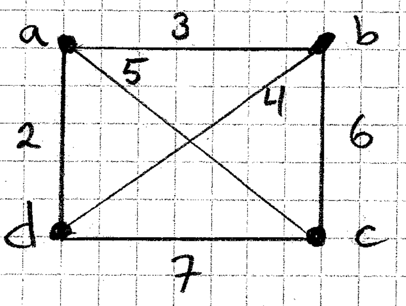
Time-complexity $O(n^2)$ where n is the number of vertices

For the traveling salesman problem there are $\frac{(n-1)!}{2}$ circuits to consider.

Hopeless!

<u>Approximation</u> <u>algorithms</u> are used.

Ex)



Start in a.

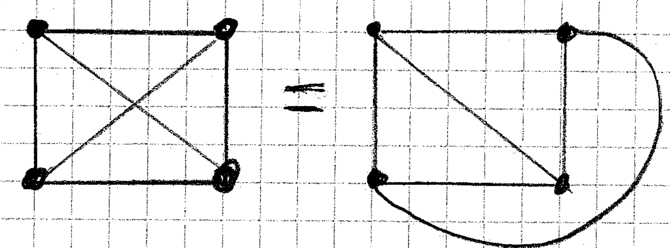
$\frac{3\cdot 2\cdot 1}{2} = 3$

cycles to consider.

Which is the shortest cycle?

9.7 Planar graphs

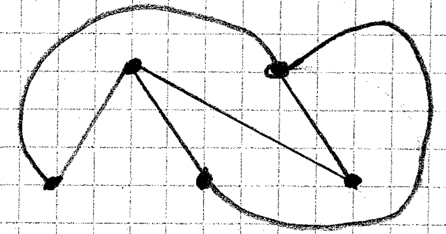
Remember K_4



K_4 is <u>planar</u>.

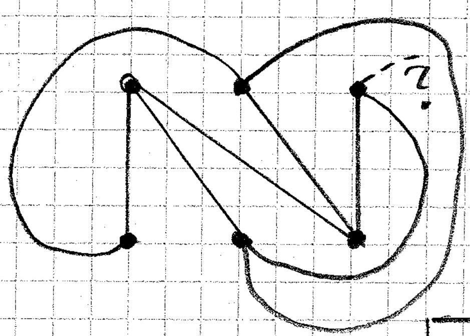
Ex) Are $K_{2,3}$ and $K_{3,3}$ planar?

$K_{2,3}$

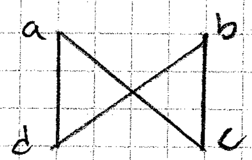


YES!

$K_{3,3}$



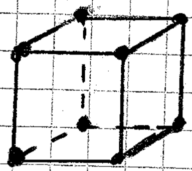
No!



Length 17

Eulers formula

Q_3

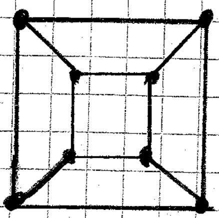


6 faces
8 vertices
12 edges

$$f = e - v + 2$$

see I. Lakatos Proofs and Refutations.

But Q_3 is planar



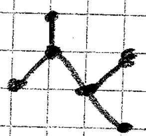
6 regions. Note one unbounded region
8 vertices
12 edges

$$\boxed{r = e - v + 2}$$

Euler's formula

Proof: Induction over regions.

$r = 1$, Then we have a tree



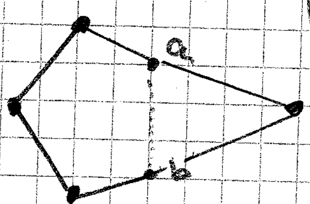
$v = e + 1$

$r = e - (e+1) + 2 = 1$ ok

Assume the formula holds for $r = k$
Consider a planar graf G with $r = k+1$
Consider an edge $\{a, b\}$ in this graf that belongs to a cycle. Take it away!

Then

G'



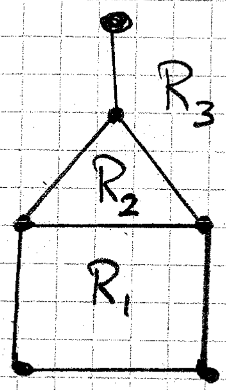
$G \to G'$
$r \to r' = r - 1$
$e \to e' = e - 1$
$v \to v' = v$

$$\underline{\underline{r - e + v}} = r' + 1 - e' - 1 + v' = \underline{\underline{2}}$$

To be planar G can not have too many edges! (keeping v fixed)

Degree of a region: $$2e = \sum_{R_i} \deg(R_i) \qquad (*)$$

Ex)



$\deg(R_1) = 4$
$\deg(R_2) = 3$
$\deg(R_3) = 7$ Note!

$4+3+7 = 14 = 2 \cdot 7 = 2 \cdot e$

The formula $(*)$ above is useful.
Assume $\deg(R_i) \geq 3$. Then

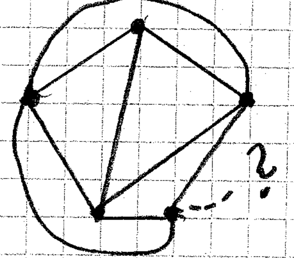
$$2e \geq 3r$$

With Euler's formula we get

$$r = e - v + 2 \leq \frac{2e}{3}$$

$$e \leq 3v - 6$$ Nescessary condition!

Ex) K_5

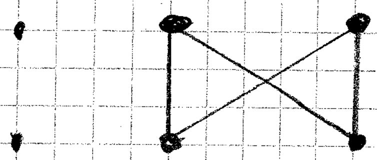


$e = 10$
$v = 5$
$3v - 6 = 9 < 10$

K_5 is not planar

Ex) $K_{3,3}$ $e = 9$, $v = 6$ $3v - 6 = 12$, $9 \leq 12$ But not sufficient!

but we can do a little bit better for $K_{3,3}$. Minimal cycle length must be 4 since it is bipartite



so $$2e \geq 4r \iff r \leq \frac{e}{2}$$

EF gives
$$r = e - v + 2 \leq \frac{e}{2}$$

$$e \leq 2v - 4$$

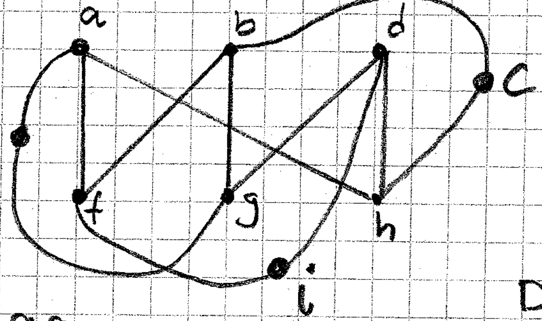
Now we can give up any hope to draw $K_{3,3}$ without crossings ☺

$e = 9$
$r = 6$ $\quad 2v - 4 = 8$

$8 < 9$. Conclusion $K_{3,3}$ is non-planar.

K_5 and $K_{3,3}$ "can be found" in all non-planar graphs!

Ex) G



is homeomorphic to $K_{3,3}$

$ae + eg \to ag$
$bc + ch \to bh$
$di + if \to df$

Delete the vertices e, i, c and we get $K_{3,3}$.

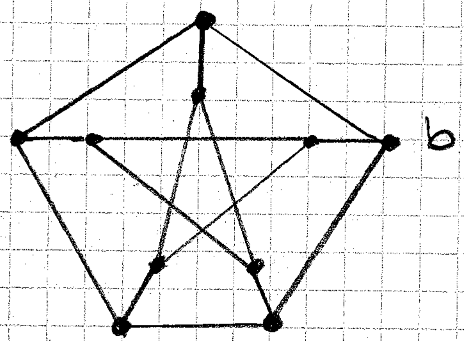
Kuratowski's theorem. (1930)

A graph G is non-planar

$$\Longleftrightarrow$$

There is a subgraph in G which is homeomorphic to $K_{3,3}$ or K_5.

Ex)



Petersen graph

If you take away vertex b and the three edges attached to it you find a graph which is homeomorphic to $K_{3,3}$.