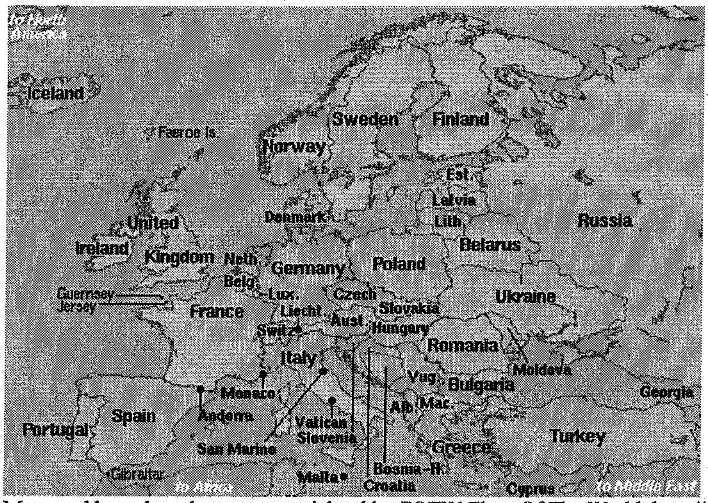
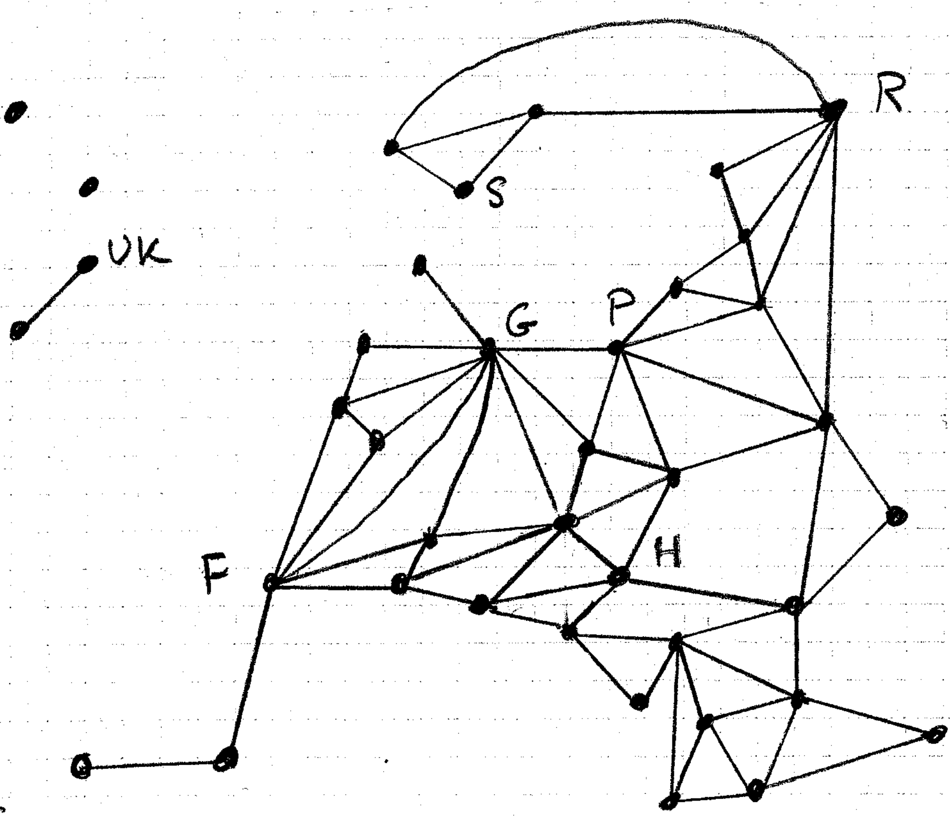
Graphs & Trees, part IV.

9.8 Graph coloring



Maps and boundary data are copyrighted by FOTW Flags Of The World, http://flagspot.net/

Which is the minimom number of colors needed to make a coloring of the map in such a way that countries with a common border have different colors?

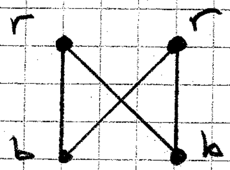
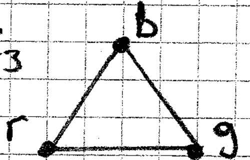
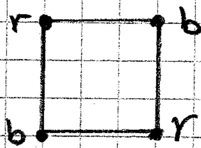
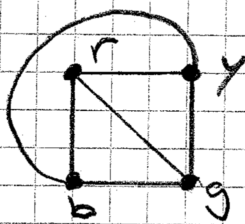
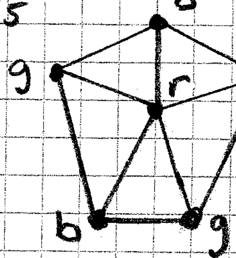


The dual graph

DEF

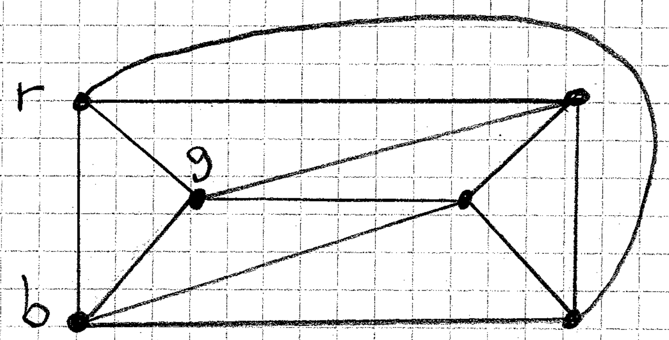
A coloring of a simple graph is the assignment of a color to each vertex of the graph so that no two adjacent vertices are assigned the same color.

The **chromatic number** for graph G is the minimum number of colors needed for such a coloring. It is denoted $\chi(G)$.

Graph G		$\chi(G)$
Bipartite	$K_{2,2}$ 	2
Paths	 P_3	2
Cycles	C_3 	3 if v odd
	C_4 	2 if v even
K_n		n, a new color for each vertex
Wheels	W_5 	4 if v even 3 if v odd

Ex) Find $\chi(G)$ for the graph

G

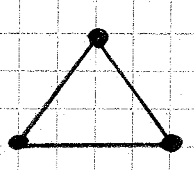


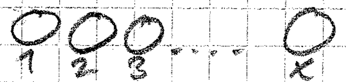
$\chi(G) \geq$?

(Not in the book)

Chromatic Polynomial $P_G(x)$ = # ways to color the graph G with x colors. $x = 1, 2, 3, 4, \ldots$

Ex)



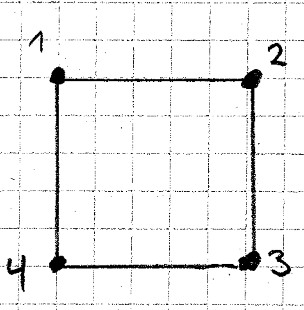


$$P_\Delta(x) = x \cdot (x-1)(x-2)$$

$$P_\Delta(1) = P_\Delta(2) = 0$$

$$P_\Delta(3) = 3! = 6$$, $P_G(x) = 0$ when $x < \chi(G)$

Ex)



$$P_\square(x) = P_\square^{(1=3)}(x) + P_\square^{(1\neq 3)}(x)$$

$$= x \cdot (x-1)(x-1) + x \cdot (x-1)(x-2)^2 =$$

$$x(x-1)(x^2-3x+3) = x^4 - 4x^3 + 6x^2 - 3x$$

$P_\square(1) = 0$, $P_\square(2) = 2 \cdot 1 \cdot 1 = 2$ OK

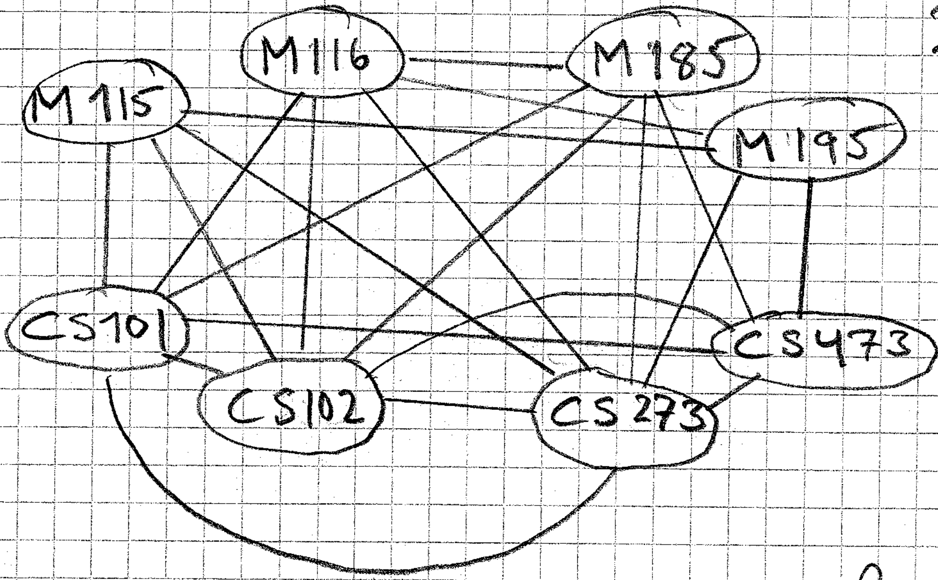
$P_\square(3) = 3 \cdot 2 \cdot 3 = 18$

Exponential worst-case time complexity for finding $\chi(G)$.

Sometimes we want to keep things apart ....

Ex)
9.8.17

G



$\chi(G) = ?$

The subgraph K_4 can be found in the CS-group. So $\chi(G) \geq ?$

Inside the M-group we see a path.
M185 is connected to all vertices in the CS-group. so $\chi(G) \geq ?$
M115 and M116 are connected to 3 of the CS-vertices.

Back to the map coloring.

Theorem: $\chi(G) \leq 4$ for planar graphs.
(1976)

Proof: Made by man + machine!

We have to be a little more modest......

<u>Theorem.</u> $\chi(G) \le 6$ for planar graphs.

<u>Proof</u>: From previous lecture we know that

$$e \le 3v - 6$$

for planar graphs.

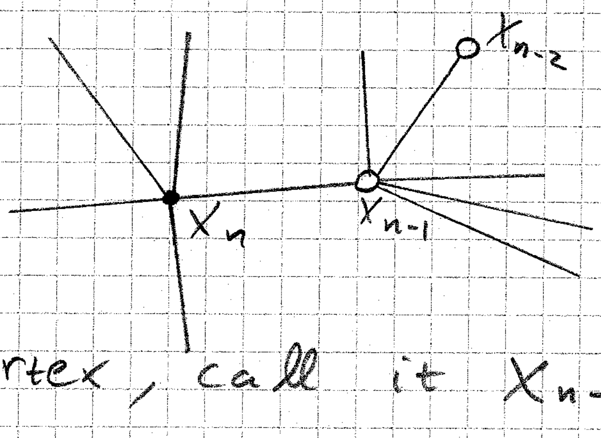
If $\deg(v) \ge 6$ for all $v \in V$ then

$$2e = \sum_{v \in V} \deg(v) \ge 6v \iff e \ge 3v.$$

Conclusion: For planar graph the minimal degree is at most 5.

Take such a vertex, call it X_n

$G - \{X_n\} = H_{n-1}$ has also such a vertex call it X_{n-1}. $H_{n-2} = H_{n-1} - \{X_{n-1}\}$ has also



such a vertex, call it X_{n-2} ....

$$X_1, X_2, X_3, X_4, X_5, X_6, X_7 \text{ - - - - - - - - } X_n$$

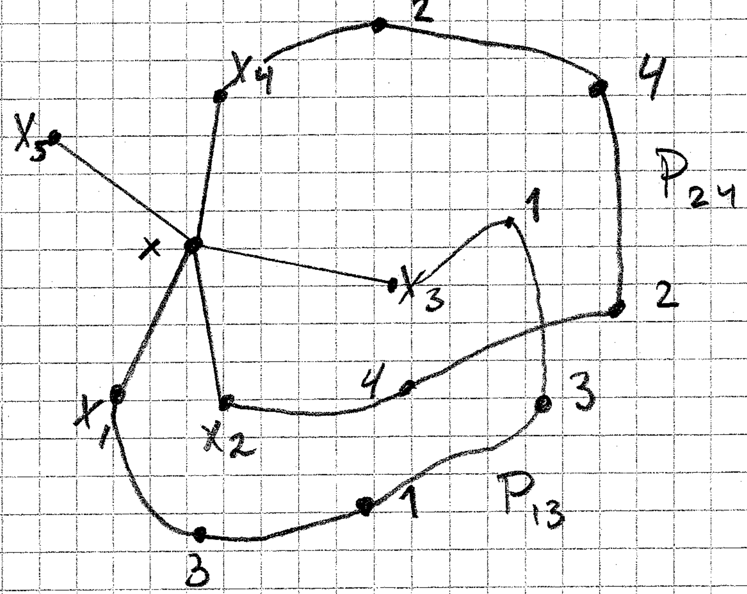
Connected to at most 5 of the previous ones. A color can be reused.

The best we can do is:

<u>Theorem</u>: $\chi(G) \leq 5$ for planar graphs.

<u>Proof</u> by <u>contradiction</u>: Let G be a 6-chromatic graph with <u>minimal</u> number of vertices. Let x be a vertex of degree at most 5 (it exists!)
Five colors is sufficent to color $H = G - \{x\}$

A part of G



x_1 has color 1, x_2 has color 2, ...,
x_5 has color 5

Let $H(i,j)$ be the subgraph of H <u>spanned</u>* by colors i and j.

Can x_1 and x_3 belong to distinct components of $H(1,3)$?

*) se 10.4

No, since we can then interchange colors 1 and 3 in the X_1-component and then color 1 is free to use for X.

Thus there is a path P_{13} from X_3 to X_1.

Using the same arguments for colors 2 and 4 we understand that there must be a path P_{24} from X_2 to X_4.

Can you now see the contradiction? Look at the drawing on previous page.

(From B. Bollobas, Modern Graph Theory)