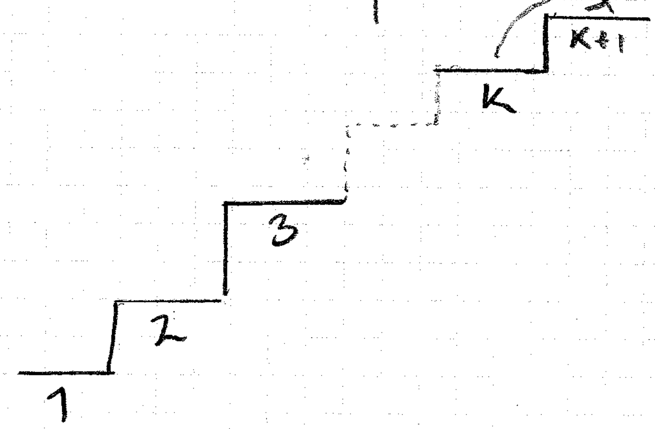
Chapter 4 – Induction and Recursion.

4.1 To prove by <u>induction</u> that $P(n)$ is true for all n do two things.

i) Verify that $P(1)$ is true

ii) Show that the conditional statement $P(k) \to P(k+1)$ is true for all positive integers k.



4.1.5) $P(n)$ is the proposition

$$1^2 + 3^2 + 5^2 + \dots (2n+1)^2 = \frac{(n+1)(2n+1)(2n+3)}{3}$$

i) Is $P(1)$ true?

LHS: $1^2 + 3^2 = 10$ Yes!

RHS: $\frac{2 \cdot 3 \cdot 5}{3} = 10$

Also $1^2 + 3^2 + 5^2 = 35 = \frac{3 \cdot 5 \cdot 7}{3}$

ii) Assume $P(k)$ is true, that is

$$1^2+3^2+5^2+\dots+(2k+1)^2=\frac{(k+1)(2k+1)(2k+3)}{3}$$

What about $P(k+1)$?

Induction step

$$\left(1^2+3^2+5^2+\dots+(2k+1)^2\right)+\left(2(k+1)+1\right)^2=$$

$$=\frac{(k+1)(2k+1)(2k+3)}{3}+(2k+3)^2$$

What do I hope for? I hope the last expression is equal to

$$\frac{(k+1+1)(2(k+1)+1)(2(k+1)+3)}{3}=$$

$\frac{(k+2)(2k+3)(2k+5)}{3}$. If so we have proven that $P(k+1)$ is true whenever $P(k)$ is true.

Let's check!

$$\frac{(k+1)(2k+1)(2k+3)}{3}+(2k+3)^2=$$

$$\frac{(2k+3)}{3}\left[(k+1)(2k+1)+3(2k+3)\right]=$$

$$\frac{(2k+3)}{3}\left[2k^2+9k+10\right]=\frac{(2k+3)}{3}(k+2)(2k+5)$$

OK!

4.1.20 $\quad P(n): \quad 3^n < n!\ , \quad n \geq 7.$

i) $P(7)$.

Is $3^7 < 7!$?

$$7! = 7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1 > (2\cdot3+1)(2\cdot3)\cdot3\cdot3\cdot3\cdot2$$
$$= 8\cdot3^5 + 4\cdot3^4 = 3^4(8\cdot3+4) > 3^4\cdot3^3 = 3^7$$

ii) Assume $3^k < k!$ and let's have a look at 3^{k+1}.

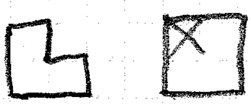
$$3^{k+1} = 3\cdot3^k \underset{\substack{\uparrow \\ \text{IA}}}{<} 3\cdot k!$$

What do you hope for? Can you finish the proof now?

An axiom for the set of positive integers is the <u>well-ordering property</u>: Every nonempty subset of positive integers has a least element. (see appendix 1).

Using this axiom we prove the validity of mathematical induction.

Don't miss the example with L-shaped figures tiling $2^m \times 2^m$ checker-boards minus one square.



4.2) Strong induction.

Here the inductive step is

$$[P(1) \land P(2) \land \ldots \land P(k)] \to P(k+1)$$

A more flexible proof technique.

Ex) $P(n)$: all integers ≥ 2 can be written as a product of primes.

$2 = 2$ $P(2)$ true!

$3 = 3$

$4 = 2 \cdot 2$

Assume $P(j)$, $j = 2, 3, \ldots k$, is true. Is then $P(k+1)$ true?

$k+1$

A prime. OK!

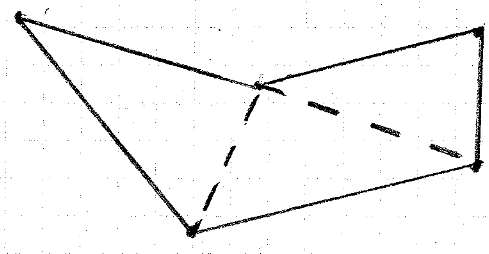
Composite number.

$k+1 = a \cdot b$

Both a and b can be written as product of primes.

OK!

Theorem 1 A simple polygon with n sides $(n \geq 3)$ can be triangulated into $n-2$ triangels



$n=5$

3 triangels.

Lemma 1: Every simple polygon has an interior diagonal.

4.3 Recursive definitions (for functions)

Ex) Fibonacci numbers

$f_0 = 0$, $f_1 = 1$

$$f_n = f_{n-1} + f_{n-2}, \quad n \geq 2$$

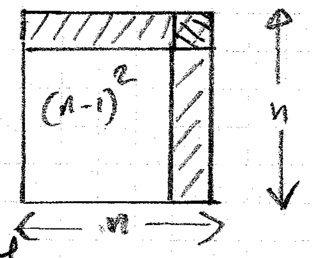
$f_{100} = ?$ (see 7.2)

$$f_{100} = f_{99} + f_{98} = 2 \cdot f_{98} + f_{97} = \cdots\cdots =$$
$$A f_2 + B f_1 \qquad A, B \text{ real constants.}$$

4.3.8d $\quad a_n = n^2 \qquad a_1 = 1,\ a_2 = 4,\ a_3 = 9,$

Give a recursive definition of a_n. $\quad a_4 = 16, \cdots\cdots$

$$a_n = (n-1)^2 + 2n - 1$$
$$a_n = a_{n-1} + 2n - 1.$$



with $a_1 = 1$ we get

$$a_2 = a_1 + 3 = 4$$
$$a_3 = a_2 + 5 = 4 + 5 = 9$$

4.3.12 Prove that

$$f_1^2 + f_2^2 + f_3^2 + \ldots f_n^2 = f_n \cdot f_{n+1}$$

$f_0 = 0$, $f_1 = 1$, $f_2 = 1$, $f_3 = 2$, $f_4 = 3$

For example $f_1^2 + f_2^2 + f_3^2 = 1 + 1 + 4 = 6 = 2 \cdot 3$

Proof: $f_{n+1} \cdot f_n = (f_n + f_{n-1}) f_n = f_n^2 + f_n \cdot f_{n-1}$

$= f_n^2 + (f_{n-1} + f_{n-2}) f_{n-1} = f_n^2 + f_{n-1}^2 + f_{n-1} \cdot f_{n-2}$

$= \cdots\cdots\cdots\cdots$