5. Counting

In how many ways can ...... ?

PRODUCT RULE. If there are n_1 ways to do the first task and n_2 ways to do the second task then there are $n_1 \cdot n_2$ ways to do the procedure. Note, the 2 tasks are separated, they can be performed independently of each other.

Ex) Flip a coin twice.
$2 \cdot 2 = 4$ outcomes. They are HH, HT, TH, TT. Doing it 10 times gives 2^{10} outcomes

SUM RULE: If the task can be done either in n_1 ways or in n_2 ways then there are $n_1 + n_2$ ways to do the task. Note the two sets are disjoint.

Ex) Flip a coin 10 times. In how many ways can you get at least two heads?

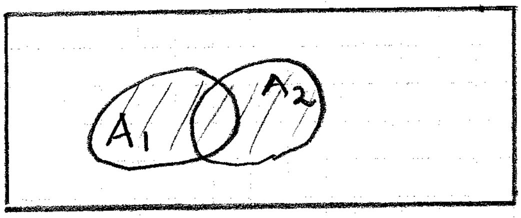
	2^{10}
0,1 H	At least 2 heads
N_1	N_2

$$N_1 + N_2 = 2^{10}$$
$$N_2 = 2^{10} - N_1$$
$$= 2^{10} - 1 - 10 =$$

$1024 - 11 = 1013$

The inclusion-exclusion principle

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$



5.1.43) Bit strings of length 10.

0001010101
1110111100
0001111100
⋮

How many bit strings of length 10 begin either with 000 or end with 00?

5.2) Pigeonhole principle

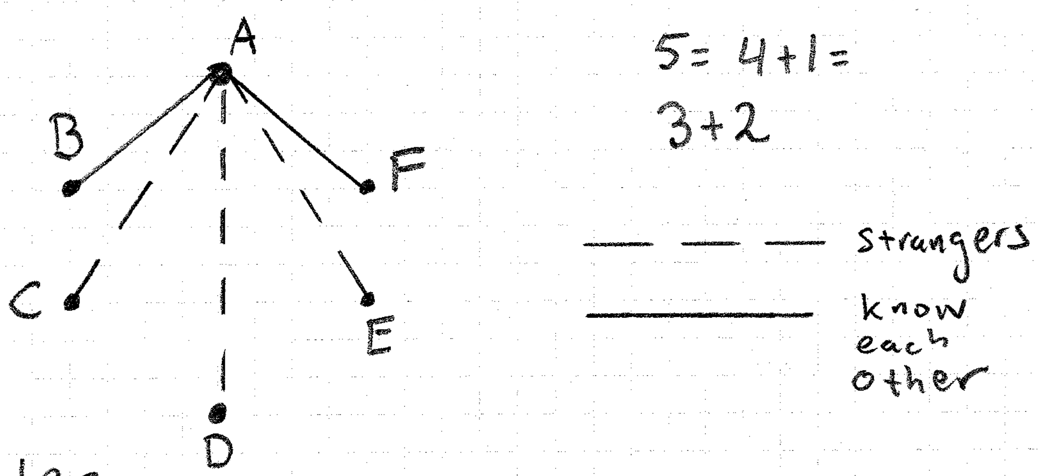
If $k+1$ objects are placed in k boxes, Then at least one box contains two or more objects.

Ex) Among 53 persons you can find at least one weekday for which at least $\lceil \frac{53}{7} \rceil = 8$ persons were born at that day.

Ex) $R(3,3)=6$. Each pair of persons are either strangers or know each other.

At a party with 6 people it is always possible to find a subgroup of 3 people that all know each other or are strangers to each other, $R(3,3)=6$

Proof



Consider know C, D, E. Two cases: C, D, E are all friends or two of them, say C and E are strangers. Can you in both cases find a triangle with sides of the same type?

$R(4,4)=18$, $43 \le R(5,5) \le 49$, $102 \le R(6,6) \le 165$

solution to 5.1.43

$|A_1| = 2^7$, $|A_2| = 2^8$

$|A_1 \cap A_2| = 2^5$

$|A_1 \cup A_2| = 2^7 + 2^8 - 2^5 = 2^5(4+8-1) =$

$11 \cdot 32 = 352.$

5.3 Permutations & Combinations

$$P(n,r) = n\cdot(n-1)\cdot(n-2)\ldots(n-r+1) = \frac{n!}{(n-r)!}$$

Ex) $n=5$ A, B, C, D, E
$r=2$

$P(5,2) = 5\cdot 4 = \frac{5!}{3!} = 20$ "words" can be formed. They are AB, AC, AD, ..., ED.

$$C(n,r) = \frac{P(n,r)}{r!} = \frac{n!}{r!(n-r)!},$$ Notation $\binom{n}{r}$ also common.

5.3.20a) How many bit strings of length 10 have exactly three 0s? _ _ 0 _ _ _ _ 0 _ 0 _

First position can be chosen in 10 ways, second in 9 and third in 8 ways, $10\cdot 9\cdot 8 = 720$ but then e.g. 379, 397, 739, 793, 937, 973 are all counted. They correspond to the same bit string!

Correct answer is $\frac{10\cdot 9\cdot 8}{6} = \frac{10!}{7!3!} = C(10,3)$
$=120$

Note $C(n,r) = C(n, n-r)$

5.4) The binomial theorem

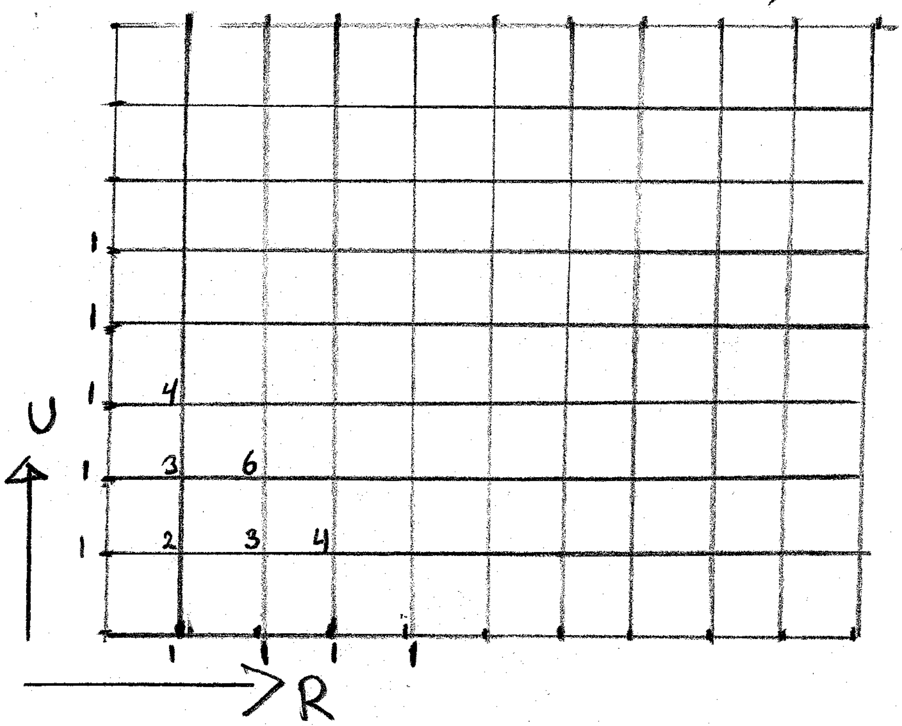
$$(x+y)^2 = \binom{2}{0}x^2y^0 + \binom{2}{1}x^1 \cdot y^1 + \binom{2}{2}x^0y^2$$

$$(x+y)^3 = \binom{3}{0}x^3y^0 + \binom{3}{1}x^2y^1 + \binom{3}{2}x^1y^2 + \binom{3}{3}x^0y^3$$

$$= x^3 + 3x^2y + 3xy^2 + y^3$$

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j$$

Pascal's identity : $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$



To go from $(0,0)$ to $(2,2)$ can be done in 6 ways. $6 = \binom{4}{2}$, it is a question about when to do the R-moves.

$6 = 3 + 3$.

From left   From below

$$\binom{4}{2} = \binom{3}{1} + \binom{3}{2}$$

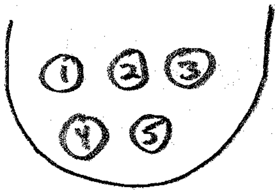
WITH RESPECT TO ORDER?

5.5)

Repetitions?	YES	NO
YES	n^r	$\binom{n+r-1}{r}$
NO	$P(n,r)$	$\binom{n}{r}$

Pick r balls from a bowl with n balls

Ex) $n=5$
$r=3$



We allow for repetitions but dont care about the order.

$X_1 = \#\ 1s$, $X_2 = \#\ 2s$ etc.

So our problem is equivalent to:

How many integer solutions are there to

$$X_1 + X_2 + X_3 + X_4 + X_5 = 3, \quad X_i \geq 0$$
$i = 1,2,3,4,5$

$3+0+0+0+0 \Leftrightarrow |||++++$
$2+1+0+0+0 \Leftrightarrow ||+|+++$
$\vdots$
$0+1+1+0+1 \Leftrightarrow +|+|++|$
$\vdots$
$0+0+0+0+3 \Leftrightarrow ++++|||$

Three sticks, 4 plus signs. 7 positions where to place the sticks? $\binom{7}{3}$ possibilities.