7. Advanced Counting Techniques, part I.

7.1 Recurrence Relations. $a_n = f(a_0, a_1, \ldots a_{n-1})$

Ex1) $\boxed{\begin{aligned} \text{IV} \rightarrow \quad & a_0 = 100 \\ \text{RR} \rightarrow \quad & a_n = 1.03 \cdot a_{n-1} \end{aligned}}$ 3% interest

Then $a_1 = 1.03 \cdot 100$

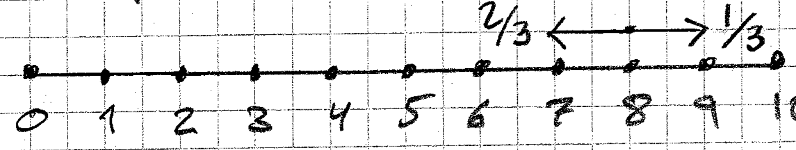
$a_2 = 1.03 \cdot 1.03 \cdot 100 = 1.03^2 \cdot 100$

$\vdots$

$a_n = 1.03^n \cdot 100$ You have 200 after 22 y.

Ex 2) You have n crowns, $0 \leq n \leq 10$.

Your opponent has $10-n$ crowns



R_n = the probability you get ruined starting with n crowns. $0 \leq R_n \leq 1$.

In every step you loose 1 crown in 2 cases out of 3

RR: $R_n = \frac{1}{3} \cdot R_{n+1} + \frac{2}{3} R_{n-1}$

Boundary values: $R_0 = 1$ and $R_{10} = 0$

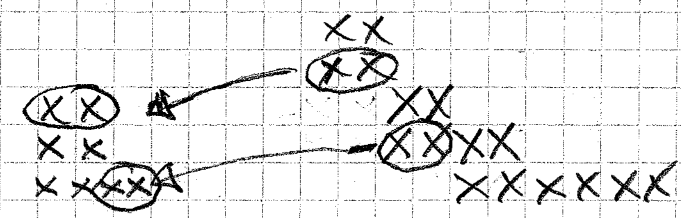
or $\qquad R_{n+1} = 3 \cdot R_n - 2 \cdot R_{n-1}$

Ex3) Leonardo from Pisa, alias Fibonacci

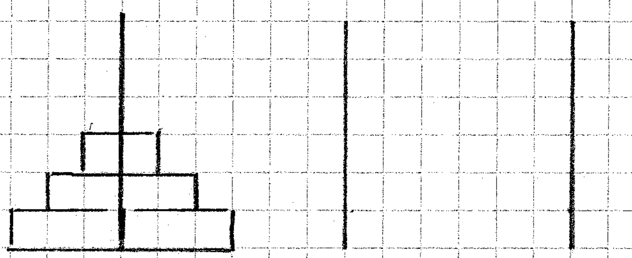
$$f_n = f_{n-1} + f_{n-2}$$
$$f_1 = f_2 = 1$$

month $\quad$ Rep. pairs $\quad$ young pairs

1
2
3
4
5



Ex4) Towers of Hanoi



Move the disks from peg 1 to 2. No disk is allowed to lie on top of a smaller disk.

$$H_1 = 1, \quad H_2 = 1 + 2 \cdot 1 = 3$$
$$H_3 = 2 \cdot 3 + 1 = 7$$

RR: $\boxed{H_n = 2\,H_{n-1} + 1}$ $\quad$ IV: $H_1 = 1$

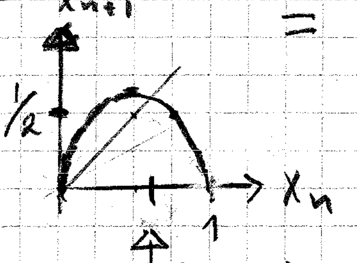
Ex 5) $\quad X_{n+1} = a\,X_n(1 - X_n) \qquad X_0 \in [0,1]$

$0 < a \leq 4$

$$= a\,(X_n - X_n^2)$$

good times! $\qquad$ shortage of food

$a = 5/2$



fixpoint $x = 3/5$

7.2 Solving Linear Recurrence Relations

$$a_n = f(a_{n-k}, \dots, a_{n-1}) =$$

$$c_1 a_{n-1} + c_2 a_{n-2} + \dots c_k a_{n-k}$$

<u>Linear</u>, <u>homogenous</u> recurrence relation of <u>degree k</u> with <u>constant coefficients</u>.

<u>Example</u>	Linear?	Homogenous	Degree
$a_n = 1.03\, a_{n-1}$	Yes	Yes	1
$R_{n+1} = 3R_n - 2R_{n-1}$	Yes	Yes	2
$H_n = 2H_{n-1} + 1$	Yes	No, due to 1.	1
$f_n = f_{n-1} + f_{n-2}$	Yes	Yes	2
$x_{n+1} = a(x_n - x_n^2)$	No, due to x_n^2.	Yes	1

Let's solve linear RR with degree 1 and 2, both homogenous and non-homogenous, and with constant coefficients.

i) $a_n = 1.03\, a_{n-1}$ generally, $a_n = r\, a_{n-1}$

$a_n = 1.03^n a_0$ then $a_n = r^n a_0$

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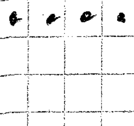
ii) $a_n = a_{n-1} + n$

$a_1 = 1$ non-homogenous RR



$a_n = C \cdot 1^n + Pn = C + Pn$

C is the homogenous solution.

P_n is the particular solution, to be found, for this particular term n.

Let's try $P_n = b_1 n + b_0$ and put it into RR:

$$b_1 n + b_0 = b_1(n-1) + b_0 + n$$

$$b_1 = n \quad ?? \quad \text{No solution!}$$

Right ansatz is $P_n = b_1 n^2 + b_0 n$

Put this into RR and show that $b_1 = b_0 = \frac{1}{2}$. Finally

$$a_n = C + \frac{n^2}{2} + \frac{n}{2}$$. Remember

$$a_1 = 1 \quad \text{so} \quad 1 = C + \frac{1}{2} + \frac{1}{2} \Leftrightarrow C = 0$$

$$a_n = \frac{(n+1)n}{2}$$

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iii)

$$R_{n+1} = 3 \cdot R_n - 2R_{n-1}$$

Assume $R_{n+1} = c \cdot r^{n+1}$ $\quad c \neq 0,\ r \neq 0$

$$c r^{n+1} = 3 c r^n - 2 c r^{n-1}$$

$$r^2 = 3r - 2$$

$$r^2 - 3r + 2 = 0$$ Characteristic equation

$$r_1 = 2, \quad r_2 = 1$$

$$R_n = b_1 2^n + b_0 1^n, \quad R_0 = 1, \quad R_{10} = 0$$

$$R_n = \frac{2^{10} - 2^n}{2^{10} - 1}$$

for example $R_8 = \frac{768}{1023}$!

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iv) 7.2.35

$a_0 = 1$
$a_1 = 4$

$$a_n = 4a_{n-1} - 3a_{n-2} + 2^n + (n+3)$$

characteristic equation

$r^2 = 4r - 3 \iff r_1 = 3,\ r_2 = 1$

Homogenous solution $b_1 3^n + b_0 1^n$

$= b_1 3^n + b_0$

particular solution for 2^n-term.

Ansatz: $c2^n$. We then get

$$c \cdot 2^n = 4c2^{n-1} - 3c2^{n-2} + 2^n$$

$$\iff$$

$$c = -4$$

particular solution for $(n+3)$-term.

Ansatz: $(d_1 n + d_0)n$. Put this into RR

$$d_1 n^2 + d_0 n = 4\left[d_1(n-1)^2 + d_0(n-1)\right]$$
$$-3\left[d_1(n-2)^2 + d_0(n-2)\right] + (n+3)$$

gives $d_1 = -1/4$ and $d_0 = -5/2$

$$a_n = b_1 3^n + b_0 - 4\cdot 2^n - \frac{n^2}{4} - \frac{5n}{2}$$

Now it is time to determine the coefficients b_1 and b_0

$$\begin{cases} a_0 = 1 = b_1 + b_0 - 4 \\ a_1 = 4 = b_1 \cdot 3 + b_0 - 8 - \frac{1}{4} - \frac{5}{2} \end{cases}$$

$$\begin{cases} b_1 = 39/8 \\ b_0 = 1/8 \end{cases}$$

Ans. $a_n = -\frac{n^2}{4} - \frac{5n}{2} + \frac{39\cdot 3^n + 1}{8} - 4\cdot 2^n$

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v) Double root

r_1^n,

$\frac{r_2^n - r_1^n}{\varepsilon}$ Two linearly independent solutions to the homogenous RR of degree two.

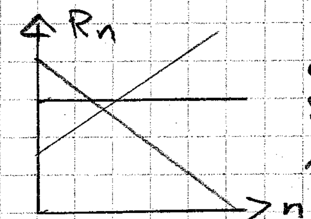
$\frac{r_2}{r_1} = 1 + \varepsilon$

Let $r_1, r_2 \to r$ then $r_1^n \to r^n$

and $\frac{r_2^n - r_1^n}{\varepsilon} = r_1^n \frac{\left(\left(\frac{r_2}{r_1}\right)^n - 1\right)}{\varepsilon} = r_1^n \left[\frac{\varepsilon n + \varepsilon^2 \ldots}{\varepsilon}\right]$

$\to r^n n$

For example $R_n = \frac{R_{n+1} + R_{n-1}}{2}$



only straight lines possible!

Characteristic equation has double root $r = 1$ so

$$R_n = b_1 1^n + b_2 n \cdot 1^n = b_1 + b_2 n$$

with $R_0 = 1$ and $R_{10} = 0$ we get $R_n = 1 - \frac{n}{10}$

 $R_8 = 0.2$ Fair game!

Ex) Complex roots

$$a_{n+2} = 2\cdot\left(a_{n+1} - a_n\right) \qquad n \geq 0$$

$a_0 = a_1 = 1$

$a_2 = 0,\ a_3 = -2,\ a_4 = -4,\ a_5 = -4 \ldots$

oscillations!

Assume $a_n = C\cdot r^n$ $\quad C$ - a constant $\neq 0$

$$C\cdot r^{n+2} = 2\left(C\cdot r^{n+1} - C r^n\right) \qquad r \neq 0$$

$$C r^n \left(r^2 - 2r + 2\right) = 0$$

$r = 1 \pm i$ No panic!

$$a_n = C_1 (1+i)^n + C_2 (1-i)^n$$

$$= C_1 \left(\sqrt{2}\, e^{i\pi/4}\right)^n + C_2 \left(\sqrt{2}\, e^{-i\pi/4}\right)^n$$

IV: $a_0 = C_1 + C_2 = 1$

$$a_1 = C_1 \sqrt{2}\, e^{i\pi/4} + C_2 \sqrt{2}\, e^{-i\pi/4} = 1$$

$$C_1 = C_2 = 1/2$$

$$a_n = (\sqrt{2})^n \frac{\left[e^{in\pi/4} + e^{-in\pi/4}\right]}{2} =$$

$$= (\sqrt{2})^n \cos\frac{n\pi}{4}$$

Back to reality!

$$a_2 = (\sqrt{2})^2 \cos\pi/2 = 0$$

$$a_3 = (\sqrt{2})^3 \cos\frac{3\pi}{4} = (\sqrt{2})^3 \cdot \frac{-1}{\sqrt{2}} = -2$$

How to make an ansatz for the particular solution

f_i	p_n
c, a constant	A, a constant
n	$A_1 n + A_0$
n^2	$A_2 n^2 + A_1 n + A_0$
$n^t, t \in \mathbf{Z}^+$	$A_t n^t + A_{t-1} n^{t-1} + \cdots + A_1 n + A_0$
$r^n, r \in \mathbf{R}$	Ar^n
$\sin \alpha n$	$A \sin \alpha n + B \cos \alpha n$
$\cos \alpha n$	$A \sin \alpha n + B \cos \alpha n$
$n^t r^n$	$r^n(A_t n^t + A_{t-1} n^{t-1} + \cdots + A_1 n + A_0)$
$r^n \sin \alpha n$	$Ar^n \sin \alpha n + Br^n \cos \alpha n$
$r^n \cos \alpha n$	$Ar^n \sin \alpha n + Br^n \cos \alpha n$

Three cases:

1) Inhomogeneous term $f(n)$ is one single term in LHS (times a constant multiple). $f(n)$ is not a solution to homogenous equation. Use second column for particular solution

2) Same as 1 but $f(n)$ is a sum of terms in first column. Add the terms for p_n.

3) If a term f_1 of f is a solution to homogenous equation then multiply the corresponding ansatz with n^s. Take integer s as small as possible.