7. Advanced Counting Techniques, part II.

7.3 Divide-and-Conquer Algorithms and Recurrence Relations.

7.3.3) $(1110)_2 \cdot (1010)_2 = 14 \cdot 10 = 140$

$1110 = 2^2 A_1 + A_0$, $A_1 = 11$ and $A_0 = 10$.

$1010 = 2^2 B_1 + B_0$, $B_1 = 10$ and $B_0 = 10$

$1110 \cdot 1010 = 2^4 A_1 B_1 + 2^2 (A_1 B_0 + B_1 A_0) + A_0 B_0$

We can reduce the number of multiplications with one!

$1110 \cdot 1010 = (2^4 + 2^2) A_1 B_1 + 2^2 ((A_1 - A_0)(B_0 - B_1))$
$+ (2^2 + 1) A_0 B_0$

$f(n)$ = # bit operations to multiply n-digit numbers.

$f(4) = 3f(2) + C \cdot 2$

generally $f(2n) = 3f(n) + C \cdot n$

$A_1 B_1 = 110$

$A_0 B_0 = 100$

$A_1 - A_0 = 1$

$B_0 - B_1 = 0$

$$
\begin{array}{r}
1110 \cdot 1010 = \quad 1100000 \\
11000 \\
10000 \\
+ \quad 100 \\
\hline
10001100
\end{array}
$$

$128 + 8 + 4 = 140$ ok.

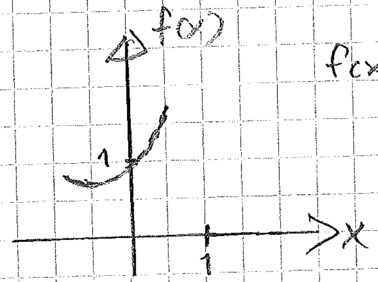
$f(n)$ is $O(n^{\log_2 3})$.

$\log_2 3 \approx 1.6$. An improvement for large n!

7.4 Generating Functions

$X^a \cdot X^b \cdot X^c = X^{a+b+c}$. Polynoms are nice can help us solving combinatorial problems!

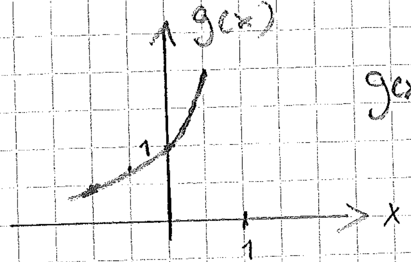
$$f(x) = \frac{x^3 - 1}{x - 1}$$



$$f(x) = 1 + 1 \cdot X + 1 \cdot X^2$$

f(x) is the <u>generating</u> <u>function</u> for the sequence 1, 1, 1, 0, 0, 0, ...

$$g(x) = \frac{1}{1-x}$$



$$g(x) = 1 + X + X^2 + X^3 + X^4 + \ldots\ldots$$

$$|x| < 1$$

g(x) is the generating function for the sequence 1, 1, 1, 1, 1, 1, ...

It is OK to derivate termwise when $|x| < 1$.

$$h(x) = g'(x) = \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \ldots$$

h(x) is the generating function for the sequence 1, 2, 3, 4, 5, ...

$$h(x) + g(x) = 2 + 3x + 4x^2 + 5X^3 + \ldots$$

$$h(x) \cdot g(x) = 1 + 3X + 6X^2 + \ldots$$

convenient notation is

$$(1-x)^{-2} = \sum_{k=0}^{\infty} \binom{-2}{k} (-x)^k$$

(extended binomial theorem)

With $\binom{-2}{k}$ we mean $\dfrac{-2 \cdot -3 \cdot -4 \cdots (-2-k+1)}{k!}$

$$= (-1)^k \cdot \frac{2 \cdot 3 \cdot 4 \cdots (2+k-1)}{k!} = (-1)^k \frac{(k+1)!}{k!} = (-1)^k k+1$$

so $\quad (1-x)^{-2} = \sum_{k=0}^{\infty} (k+1) x^k$

Get used to table 1 !

Using GF to solve counting problems.

3 children eating 17 cakes!

Ex) $\quad e_1 + e_2 + e_3 = 17 \qquad (*)$

$$2 \le e_1 \le 5, \quad 3 \le e_2 \le 6, \quad 4 \le e_3 \le 7$$

How many integer solutions are there to $*$ with these restrictions?

We are seeking the coefficient in front of x^{17} for the following function

$$G(x) = \underbrace{(x^2+x^3+x^4+x^5)}_{\text{child 1}}\underbrace{(x^3+x^4+x^5+x^6)}_{\text{child 2}}\underbrace{(x^4+x^5+x^6+x^7)}_{\text{child 3}}$$

$$= x^9 + \ldots\ldots + 3x^{17} + x^{18}$$

Let's "zip": $G(x) = x^2 \cdot x^3 \cdot x^4 \cdot (1+x+x^2+x^3)^3$
$= x^9 \dfrac{(1-x^4)^3}{(1-x)^3}$

Let's "unzip" using Table 1

$G(x) = x^9 \cdot (1-3x^4+3x^8-x^{12})\left(1+3x+\ldots\binom{k+2}{k}x^k+\ldots\right)$

How to get x^{17}?

1) $x^9 \cdot 1 \cdot x^8$. The coefficient is $\binom{10}{8}$.

2) $x^9 \cdot x^4 \; x^4$. The coefficient is $-3\cdot\binom{6}{4}$.

3) $x^9 \cdot x^8 \cdot 1$. The coefficient is 3

So the coefficient in front of x^{17} is $\binom{10}{8} - 3\binom{6}{4} + 3 = 45 - 3\cdot 15 + 3 = 3$

Ex) $e_1 + e_2 + e_3 = 17$, $e_i \geq 0$ $i = 1,2,3$

17 sticks 2 plus signs. $\binom{17+2}{17} = \binom{19}{2} = 171$

With GF:

$G(x) = (1+x+x^2+\ldots)(1+x+x^2+\ldots)(1+x+x^2+\ldots)$

$= \dfrac{1}{(1-x)^3} \underset{\uparrow}{=} \sum_{k=0}^{\infty} \binom{k+2}{k} x^k$

see table 1

Coefficient in front of x^{17} is $\binom{19}{17}$. OK

7.4.33 Solve the RR

$$(*) \qquad a_k = 3a_{k-1} + 2 \;, \; a_0 = 1$$

using GF.

$G(x) = \sum_{k=0}^{\infty} a_k x^k$ denotes the GF for the sequence $a_0, a_1, a_2, \ldots$

we note that $x\,G(x) = \sum_{k=0}^{\infty} a_k x^{k+1} = \sum_{k=1}^{\infty} a_{k-1} x^k$

Step 1) Multiply $(*)$ x^k

$$a_k x^k = 3a_{k-1} x^k + 2x^k$$

Step 2) Sum from $k=1$ to ∞

$$\sum_{k=1}^{\infty} a_k x^k = 3\sum_{k=1}^{\infty} a_{k-1} x^k + 2\sum_{k=1}^{\infty} x^k$$

Using the expressions for $G(x)$ and $x\,G(x)$ above we get

$$G(x) - a_0 = 3x\,G(x) + \frac{2x}{1-x}$$

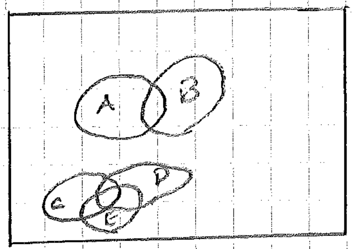
$a_0 = 1$ so $\quad G(x) = \dfrac{1+x}{(1-x)(1-3x)} = \dfrac{2}{1-3x} - \dfrac{1}{1-x}$

"unzip" $\quad = 2\sum_{k=0}^{\infty} 3^k x^k - \sum_{k=0}^{\infty} x^k$ so

$$\boxed{a_k = 2 \cdot 3^k - 1}$$

7.5 - 7.6

Inclusion - Exclusion



$|A \cup B| = |A| + |B| - |A \cap B|$

$$|C \cup D \cup E| = |C| + |D| + |E| - |C \cap D| - |C \cap E| - |D \cap E| + |C \cap D \cap E|$$

This term was added 3 times then subtracted 3 times so we have to add it once.

$$|A_1 \cup A_2 \cup A_3 \cup A_4| = \sum_{i=1}^{4} |A_i| - \sum_{1 \le i < j \le 4} |A_i \cap A_j| + \sum_{1 \le i < j < k \le 4} |A_i \cap A_j \cap A_k| - |A_1 \cap A_2 \cap A_3 \cap A_4|$$

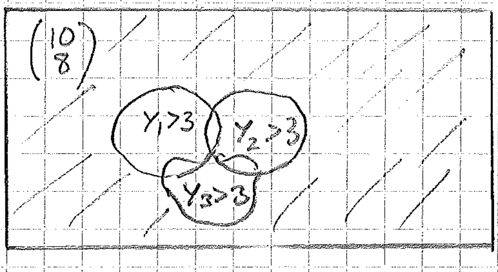
The elements that are common for all 4 sets are counted $4 - 6 + 4 - 1 = 1$ time. ok!

EX) Let's have a look at example at page 3 again

$$e_1 = 2 + Y_1, \quad e_2 = 3 + Y_2, \quad e_3 = 4 + Y_3$$

so

$$Y_1 + Y_2 + Y_3 = 8, \quad 0 \le Y_i \le 3, \quad i = 1, 2, 3$$



Without any upper restrictions there are $\binom{10}{8}$ solutions. 8 sticks and 2 plus signs. We are seeking the number of elements in shaded region. The number of elements in white region can be calculated with inclusion-exclusion.

$N(Y_1>3)$: $Y_1 = 4+U_1$

$U_1+Y_2+Y_3=4$ $\quad\binom{6}{4}$ possibilities.

$N(Y_1>3 \text{ and } Y_2>3)$: $Y_1=4+U_1$

$Y_2=4+U_2$

$U_1+U_2+Y_3=0$ $\quad$ 1 possibility.

Finally

$$N=\binom{10}{8}-\left[3\cdot\binom{6}{4}-3\cdot 1+0\right]$$

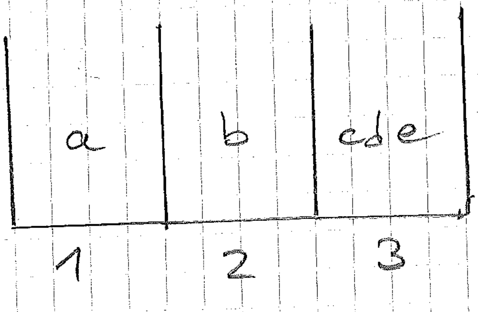
$$=45-3\cdot 15+3=3.$$

Number of onto functions

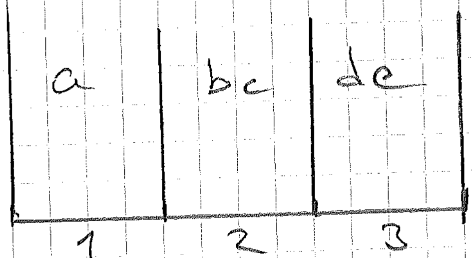
5 elements a, b, c, d, e shall be distributed into 3 containers in such a way that no container is empty

$$5 = 1+1+3 = 1+2+2$$

$1+1+3$



$1+2+2$

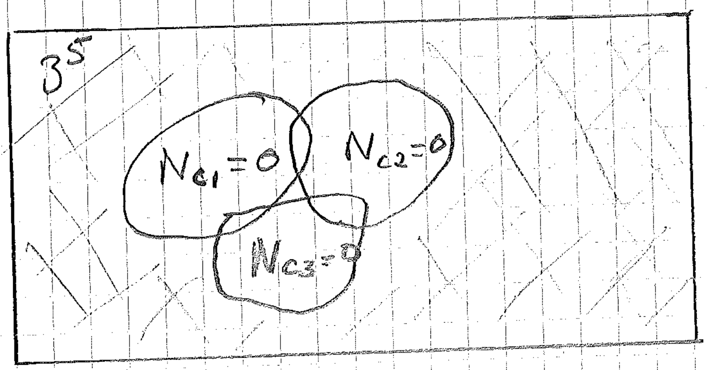


$\binom{5}{3} \cdot 3! = 60$

$5 \cdot 6 \cdot 3 = 90$

$$90 + 60 = 150.$$

with I-E



$N_{c_1} = 0$: 2^5 such functions

$N_{c_1} = N_{c_2} = 0$: $1^5 = 1$ such functions

\# onto functions are =

$$3^5 - 3 \cdot 2^5 + 3 \cdot 1^5 - 0 =$$

$$243 - 96 + 3 = 150$$