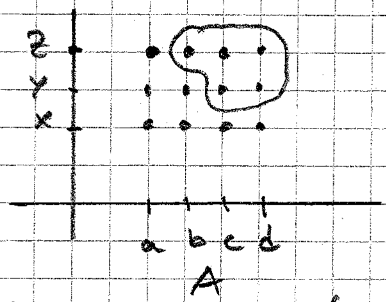
8. Relations, part I

child, partner, parent ......
Useful when handling information in databases.

8.1 DEF: A and B are sets.
A <u>binary</u> <u>relation</u> from A to B is a subset of $A \times B$

Ex)



$R = \{(b,z), (c,y), (c,z), (d,y), (d,z)\}$

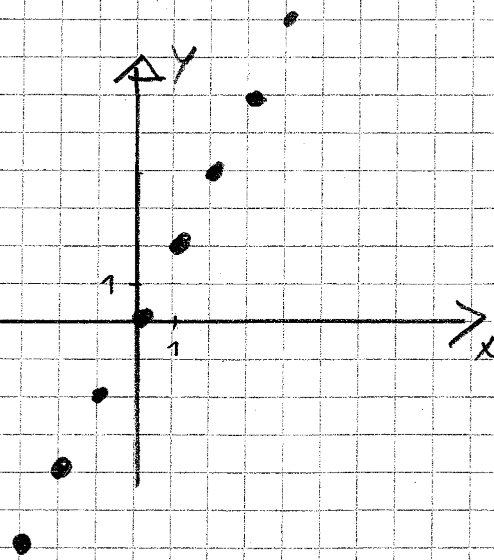
$b R z$ since $(b,z) \in R$

$a \not R x$ since $(a,x) \notin R$

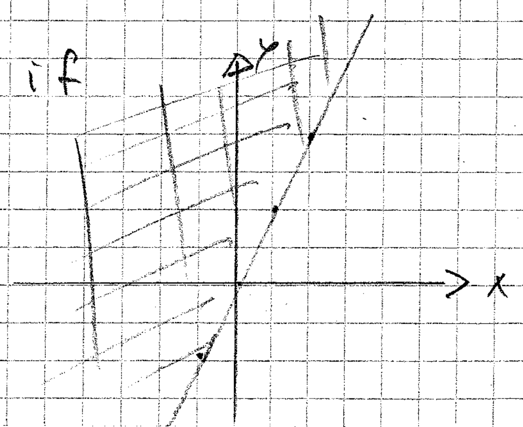
Ex) $A = B = \mathbb{Z}$

$x R_1 y$ if $y = 2x$

R_1 is a function



$x R_2 y$ if
$y > 2x$



A relation on a set A is a relation from A to A.

Ex) How many relations are there on $\{a, b, c, d\} \times \{x, y, z\}$?

Ans. $4 \cdot 3 = 12$ elements. Two choices for each point, include or not include in R. It's like tossing a coin! $2^{12} = 4096$ different relations.

Four properties that are used to classify relations on a set A:

i) R is reflexive if $(a,a) \in R$ for every $a \in A$. For example to have the same mother and father.

ii) R is symmetric if $(b,a) \in R$ whenever $(a,b) \in R$, for all $a, b \in A$. For example to vote on the same party.

iii) If $(a,b) \in R \wedge (b,a) \in R \longrightarrow a = b$ for all $a, b \in A$ then R is antisymmetric. Like $\geq$

iv) If whenever

$$(a,b) \in R \wedge (b,c) \in R$$
$$\rightarrow (a,c) \in R$$

for all $a,b,c \in A$ then R is said to be <u>transitive</u>.

8.1.7 Relation on $\mathbb{Z}$

	$(x,y) \in R$ iff	reflexive	symm.	antisymm	transitive
b)	$xy \geq 1$	No	Yes	No	Yes 1)
d)	$x \equiv y \mod 7$	Yes	Yes	No	Yes 2)
h)	$x \geq y^2$	No	No	Yes	Yes 3)

1) $ab \geq 1$ and $bc \geq 1$. a, b, c have the same sign. No one is zero

$ab \cdot bc \geq 1$

$ac \cdot b^2 \geq 1$ so $ac \geq 1$.

2) $a - b = k \cdot 7$

$b - c = l \cdot 7$

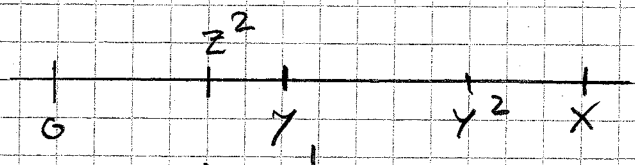
So $a - b + b - c = a - c$

$(k+l) \cdot 7$

3) $x \geq y^2$

$y \geq z^2$

Is $x \geq z^2$?



Yes!

Combining Relations. $R_1 \cup R_2$, $R_1 \cap R_2$

$R_1 - R_2$ and so on.

Composite relation: A, B, C sets

R a subser of $A \times B$
S a subset of $B \times C$

The composite relation, $S \circ R$, consits of all orded pairs (a, c) where $a \in A$ and $c \in C$ for which there exists an element $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$

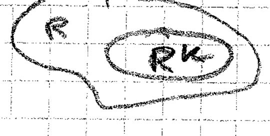
Notation $R \circ R = R^2$, $R^2 \circ R = R^3$ etc.

Ex) Parent relation. aRb if a is a parent of b. bRc if b is a parent to c.
$(a, c) \in R^2$ when.........

Theorem. The relation R on a set A is transitive $\Longleftrightarrow$ $R^n \subseteq R$
$n = 1, 2, 3 \ldots$

$\Leftarrow$) If $(a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R^2$ by definition. Since $R^2 \subseteq R$ then follows that $(a, c) \in R$
So R is transitive.

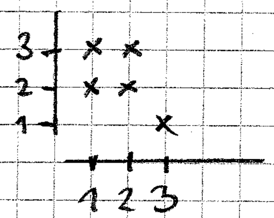
$\Rightarrow$ Use mathematical induction. Assume $R^k \subseteq R$ for some k. Show that $R^{k+1} \subseteq R$ using the transitivity
Assume $(a, b) \in R^{k+1}$ .... $(a, b) \in R$.

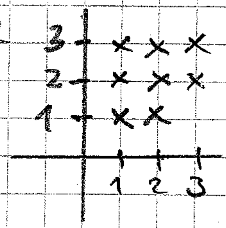


Ex) $A = \{1, 2, 3\}$

$R = \{(1,2), (1,3), (2,2), (2,3) (3,1)\}$

$R^2 = \{(1,2), (1,3), (1,1), (2,2), (2,3),$
$(2,1), (3,2), (3,3)\}$

R 

R^2 

Is R transitive ?

8.2 n-ary relations. Important for Databases!

$A_1, A_2, A_3, \ldots A_n$ are sets

A subset of $A_1 \times A_2 \times A_3 \ldots \times A_n$ is called an <u>n-ary relation.</u>

$A_1, A_2 \ldots A_n$ are called <u>domains</u> and n is the <u>degree</u>.

Think of trains. Train number, departure time, track, destination, ...

8.3 Representing relations.

Binary relations from now on.

Let us take the example above

$R = \{(1,2), (1,3), (2,2), (2,3), (3,1)\}$

The <u>zero-one matrix</u> for this relation is

$$M_R = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

An element m_{ij} of M_R is 1 if iRj and 0 if $i \not R j$. m_{ij} is the element at i:th row and j:th column. Easy to check reflexivity, anti-symmetry/symmetry.

What about transitivity?

$$M_{R^2} = M_R \odot M_R = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \odot \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

For example $(M_{R^2})_{23} = (0,1,1)\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 1 \cdot 0 + 1 \cdot 1 + 0 \cdot 1 = 1$

Note!

$2R2$ and $2R3$ so $2R^2 3$.

See 3.8 about matrices.

Since $M_{R^2} \vee M_R = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ we see, by comparing wit M_R, that R is not transitive.

The <u>digraph</u> for R is