8. Relations, part II.

8.4 Closures of Relations.

How to enlarge R to get a desired property? (Note minimal enlargement)

We take our example from part I again

Ex) $R = \{(1,2), (1,3), (2,2), (2,3), (3,1)\}$

$$M_R = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Reflexive closure of R =

$\{(1,1), (1,2), (1,3), (2,2), (2,3), (3,1), (3,3)\}$

include 1's in the diagonal of M_R

Symmetric closure of R =

$\{(1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2)\}$

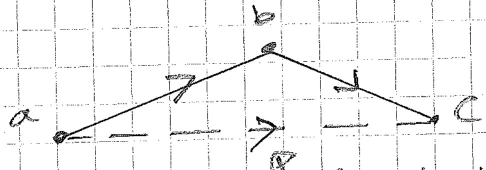
or $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

What about transitivity?

If $(a,b) \in R$ and $(b,c) \in R \Rightarrow (a,c) \in R$ then R is transitive

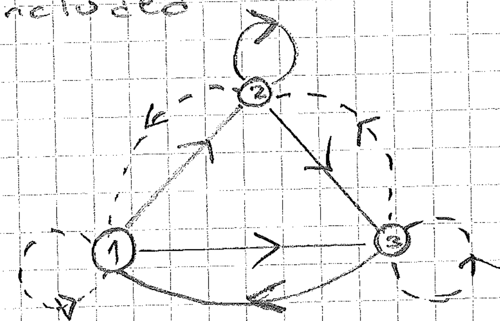
The <u>transitive closure</u> is the minimum number of pairs (a,b) to be added to R to get the transitivity property.

A path of length 2 $(a,b),(b,c) \to$ a path of length 1 (a,c).



must be included

In our example:



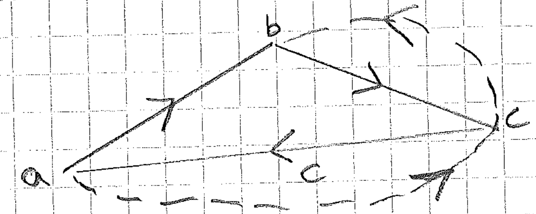
Dashed curves must be included in the transitive closure.

$$M_R = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \qquad M_{R^2} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\underbrace{M_R \vee M_{R^2} \vee M_{R^3}}_{} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \qquad M_{R^3} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Zero-one matrix for transitivity closure when $|A| = 3$. Or $R^* = R \cup R^2 \cup R^3$

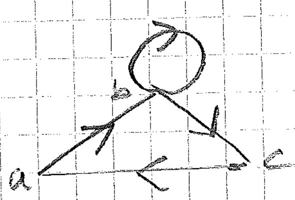
Why R^3?



Not enough! (a,a), (b,b), (c,c) must be included.

(a,a), (b,b), (c,c) are elements of R^3.

Why not R^4?



can be replaced by

8.5 Equivalence relations

are reflexive, symmetric and transitive.

Ex) on $\mathbb{Z}$. xRy iff $(x-y) = k \cdot 7$, $k \in \mathbb{Z}$.

So $8R15$ or $8 \sim 15$

$[0]_R = \{\ldots, -14, -7, 0, 7, 14, \ldots\}$ – the equivalence class of 0.

$$\mathbb{Z} = [0]_R \cup [1]_R \cup [2]_R \cup [3]_R \cup [4]_R \cup [5]_R \cup [6]_R$$

$\mathbb{Z}$

0	1	2	3	4	5	6
⋮	⋮		⋮	⋮	⋮	⋮
28	64	51	-11	-59	26	97
⋮	⋮		⋮	⋮		⋮

A partition of $\mathbb{Z}$.

Th 1: Three equivalent statements for an equivalence relation R on a set A.

i) aRb ii) $[a] = [b]$ iii) $[a] \cap [b] \neq \emptyset$

i → ii → iii → i

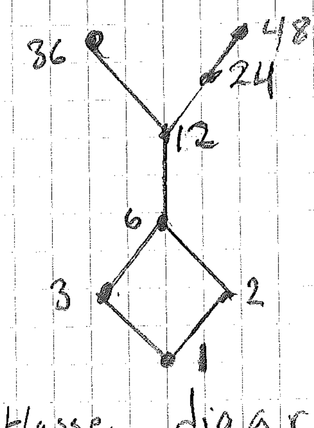
i → ii: aRb. If we can show $[a] \subseteq [b]$ and $[b] \subseteq [a]$ then $[a] = [b]$. Take $c \in [a]$.
then aRc. So cRa and therefore cRb so $c \in [b]$. We have shown that $[a] \subseteq [b]$...

8.6 Partial Orderings are relations that are
i) reflexive,
ii) anti-symmetric
iii) transitive.

Think of $\geq$ and $\supseteq$.
General symbol is $\preccurlyeq$
poset is short for Partial Orded set.

Ex) 8.6.23 c $S = \{1, 2, 3, 6, 12, 24, 36, 48\}$
and R is the divides relation.
$aRb \Leftrightarrow a|b$ that is there is an integer k such that $b = ka$.
Check i) aRa since $a = 1 \cdot a$
ii) $b = ka \wedge a = l \cdot b$ so
$b = klb$. $kl = 1$. On the positive integers $k = l = 1$
iii) $b = ka \wedge c = lb$
$c = lb = l \cdot ka$ so
aRc if aRb and bRc.

$$M_R = \begin{bmatrix} 1&1&1&1&1&1&1&1 \\ 0&1&0&1&1&1&1&1 \\ 0&0&1&1&1&1&1&1 \\ 0&0&0&1&1&1&1&1 \\ 0&0&0&0&1&1&1&1 \\ 0&0&0&0&0&1&0&1 \\ 0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&1 \end{bmatrix}$$



Hasse diagram

No loops!
No arrows!

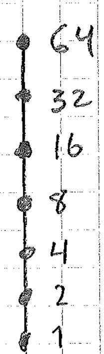
2 and 6 are <u>comparable</u>,
24 and 36 are <u>incomparable</u>

36 and 48 are <u>maximal elements</u> since they are not related to any elements in S.

1 is a <u>minimal element</u>. No element in S is related to 1.

d) $S = \{1, 2, 4, 8, 16, 32, 64\}$ with divides relation.

This is a <u>totally orded set</u> since every two elements are comparable.



← <u>greatest element</u>

← <u>least element</u>

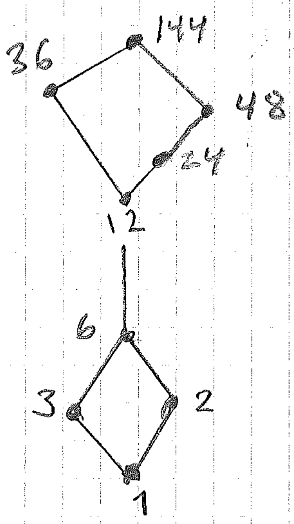
They are unique!

$aRa' \wedge a'Ra \Rightarrow a = a'$

A <u>well-orded set</u> is a poset such that $\leqslant$ is a total ordering and every non-empty subset of S has a least element. Think of $(\mathbb{Z}^+, \leq)$.

<u>Lattices</u>. A poset in which every pair of elements have <u>a least upper bound</u> and <u>a greatest lower bound</u> is called a lattice.

Ex)



glb $\{24, 36\} = 12$

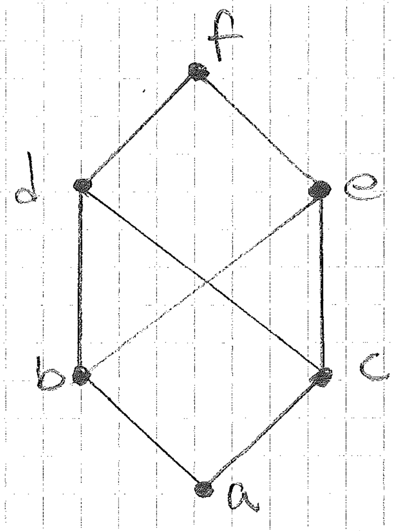
lb $\{24, 36\} = 12, 6, 3, 2, 1$

but $6, 3, 2, 1$ are all related to 12

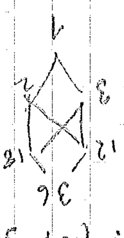
lub $\{2, 3\} = 6$. 6 is related to all other ub $\{2, 3\}$.

Checking every pair we can convince ourselves that this is a lattice.

This is not a lattice!



Why? Try to find a pair without a lub.
Give positive integers to a, b, c, ..., f so this is a Hasse diagram with the divides relation!



Ans. $\{b, c\}$