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Mathematics
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Problems

1. Consider the statement $p \to q$ where the propositions are $p : n^3+5$ *is odd* and $q : n$ *is even*. Here n is an integer. Prove it by contraposition.

2. Logic can be fun! As demonstrated by Raymond Smullyan (1919-2017) in his many books. Here is an example. You come to an island where two creatures live. The *knights* always tell the truth and the *knaves* always lie. You meet A and B on the the road and they tell you:

   A: B is a knight.
   B: The two of us are opposite types.

   Is it possible to determine the type of A and B?

3. The domain is the integers $\mathbb{Z}$. Determine the truth value of
   a)$\forall\ n\ \exists\ m\ (n^2 < m)$,
   b)$\exists\ n\ \forall\ m\ (n < m^2)$,
   c)$\forall\ n\ \exists\ m\ (n + m = 0)$.

4. Check the $3x + 1$ conjecture for starting value 15. See page 112 in eigth edition or read about it on the internet. It is also called the Collatz conjecture.

5. True or false?
   a)$x \in \{x\}$,
   b)$\{x\} \subseteq \{x\}$,
   c)$\{x\} \in \{x\}$.

6. The set A has five elements, $|A| = 5$. The set B has seven elements, $|B| = 7$. Determine $|A \times B|$.

7. A and B are sets. Is $A - B = A \cap \overline{B}$? Draw the Venn diagram.

8. f is the function $f(m, n) = m^2 - n^2 = (m - n)(m + n)$ from $\mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$. Is f onto (surjective)? What can you say about the range of f?

9. In base two we consider two numbers, 111 and 101. Do the multipliction of the two numbers with standard algorithm. Convert to base 10 and check your result. The number of bit operations for multiplying two n digit numbers is of which order?

Answers or hints

1. If n is odd then $n = 2k + 1$ where k is an integer. Then $n^3 = (2k + 1)^3 = 8k^3 + \cdots + 1$ so n^3 is odd and therfore is $n^3 + 5$ even.

2. Consider the four different cases. Both are knaves, that is the only possibility.

3. All are true.

4. $15 \to 46 \to 23 \to 70 \to 35 \to 106 \to \cdots \to 4 \to 2 \to 1 \to 4 \cdots$

5. a) True, b) True, c) False.

6. 35.

7. Yes. The elements belong to set A but not to set B.

8. Not onto, $f(m, n) \neq 2$ for all pairs m, n. For an odd number o choose $n = (o - 1)/2,\ m = (o + 1)/2$. In a similar manner the even numbers divisible with 4 can be reached. However $2, 6, 10, 14 \cdots$ are not in the range since $m - n$ even implies $m + n$ even.

9. The multiplication gives $(100011)_2 = 2^5 + 2^1 + 2^0 = (35)_{10}$. $(111)_2 = (7)_{10}$ and $(101)_2 = (5)_{10}$. With standard algorithm the number of bit operations grow like n^2, it is $\mathcal{O}(n^2)$.