Linnaeus University
Mathematics
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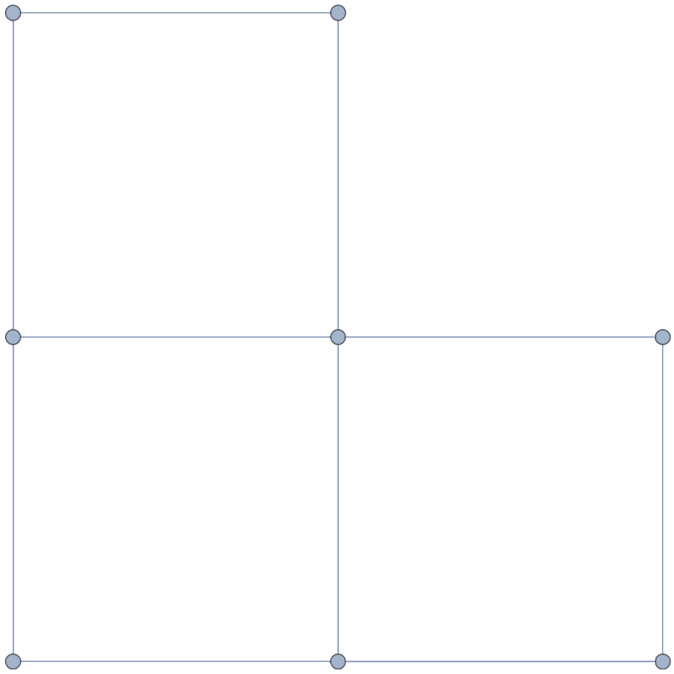
Problems

1. Prove by induction that $3^n < n!$ when the integer $n \geq 7$.

2. Prove that L-shaped figures, like the one below, tile the checkerboards of size $2^n \times 2^n$ if *one* square is excluded. The squares of the board and the L-figure have the same size. You can of course rotate the L-figures. It is possible to do it for any choice of excluded square. See example 14 on page 348 in 8:th edition. For $n = 1$ you need only one L-figure. For $n = 2$ you exclude one square in one of the four 2 times 2 boxes. How to put then the first figure? Use induction!



3. $f(n+1) = f(n) - f(n-1), \quad n \geq 1$. If $f(0) = f(1) = 1$ calculate $f(2), f(3), f(4)$ and $f(5)$.

4. How many bit strings of length seven either begin with 00 or end with 010?

5. Prove that the Ramsey number $R(3,3) = 6$. See page 3 in my lecture 5.

6. Find the number of integer solutions to

$$x_1 + x_2 + x_3 = 3$$

   when the integers $x_i \geq 0$, $i = 1, 2, 3$. Use stick and + sign method!

7. Find the coefficient of $x^3y^2z^5$ in

$$(x+y+z)^{10}.$$

8. A particle moves from (0,0) to (7,4). Allowed motion is

   $R : (x, y) \to (x + 1, y)$
   $U : (x, y) \to (x, y + 1)$
   $D : (x, y) \to (x + 1, y + 1)$

   How many paths are there? Hint: use the sum rule over the number of diagonal moves D.

Answers or hints

1. $7! - 3^7 = 2853 > 0$. Assume for some $n = k$ that $3^k < k!$. Since $(k+1)! = (k+1) \times k!$ and $3^{k+1} = 3 \times 3^k$ and furthermore $(k+1) > 3$ the inequality must hold also for $n = k+1$.

2. Consider the $2^n \times 2^n$ board. Exclude one square in one of the four $2^{n-1} \times 2^{n-1}$ boxes. Put your first L at the center in such a way that now one box is excluded in each of the four $2^{n-1} \times 2^{n-1}$ boxes. Time to use induction for each of the four boxes.

3. The recurrence relation gives the sequence $1, 1, 0, -1, -1, 0, 1, 1, \cdots$.

4. $2^5 + 2^4 - 2^2 = 44$. This is the inclusion-exclusion principle in its simplest form. We have to exclude the strings $00 \cdots 010$ once since we have counted them twice before.

5. $R(3,3) = 6$ means that in any group of six people it is always possible to find a subgroup of three where all 3 are friends to each other (on Facebook for example) or strangers to each other. Consider the complete graph with 6 vertices, that is there is an edge between any pair of vertices. Now you color the edges with red pen (friends)or blue pen (strangers). Ramsey's number tells us that a red or blue triangle must appear in the graph. Condsider vertex 1, it is connected with five other vertices. Say three of the edges from 1 is red, say to vertex 3,4 and 5. Consider now the edges between 3, 4 and 5. They can be all blue or otherwise at least one of them is red. Can you finish the proof now? Don't forget to consider the cases with 0,1,2,4 and 5 red edges going out from vertex 1.

6. $\binom{5}{2}$=10. Since we have so small numbers we can also use the sum rule. Three of the type 3+0+0, six of the type 0+2+1 and one 1+1+1. 3+6+1=10.

7. You can use the binomial theorem twice, first for $((x+y)+z)^{10}$ and then for $(x+y)^5$. We get $\binom{10}{5}\binom{5}{3} = \frac{10!}{5!3!2!}$. We can also think in the following way, ten letters can be permutated in 10! ways. However we have to divide with the number of permutations among the x, y, z.

8. $\frac{11!}{7!4!} + \frac{10!}{6!3!} + \frac{9!}{5!2!2!} + \frac{8!}{4!3!} + \frac{7!}{3!4!} = 2241$. Here I have used the sum rule over the number of diagonal moves. Going from 0 D-moves to the left to 4 D-moves to the right.