Linnaeus University
Mathematics
Hans Frisk

Problems counting techniques

1. Consider passwords with four symbols. Allowed symbols are the 26 letters in the English alphabet and the ten digits 0 to 9. We put the following restrictions on the passwords:

   i) There must be both letters and digits.
   ii) There must be both upper- and lower-case letters.
   iii) A digit or an upper case letter or a lower case letter can only appear once in the password.

   All three conditions must be fulfilled. So K8kz, 89Lj and 4ZLb are examples of allowed passwords while abcd, 3458, 23TT and KjK4 are forbidden. How many such passwords can be constructed? Use sum rule over the number of digits it the password.

2. How many words can be formed by R,R,B,B,H,H,Q, K? A very, very much harder question is to determine the number of words if we have the two following restrictions:

   i) K must be in between the two Rs.
   ii) One B must be in an odd position and the other in an even position.

   So for example BQRHKBRH is allowed. One B is in position 1 and the other in position 6. Use the sum rule for the position of K. It can not be in position 1 or 8. Put it first in position 2, then decide where to put your two R's, continue with the B's and finish with Q,H and H. K in position 7 gives the same number. The number of words when K is in position 3 or 4 are harder to calculate.

3. How many integer solutions are there to the problem

   $$x_1 + x_2 + x_3 = 8$$

   if $x_i \geq 0, i = 1, 2, 3$?

4. How many integer solutions are there to the problem

   $$x_1 + x_2 + x_3 = 28 \quad ?$$

   if $4 \leq x_1 \leq 9,\ \ 5 \leq x_2 \leq 10,\ \ 6 \leq x_3 \leq 11$? Solve the problem with inclusion and exclusion.

5. Just to remind you about *bit strings*: There are two bit strings of length one, 0 and 1. An example of a bit string of length four is 1100.

   (a) How many are the bit strings of length three?
   (b) How many are the functions from the bit strings of length three to the bit strings of length one?
   (c) How many of these functions in b are not onto (surjective)?
   (d) How many of these functions in b are sensitive to changes in all three positions? Here you can use inclusion-exclusion. Exclude the functions that are insensitive to changes in the first position, $f(0XY) = f(1XY)$ for all four possible choices of X, Y. Insensitive in for example position 1 and 3 means $f(0X0) = f(0X1) = f(1X0) = f(1X1)$ for $X = 0, 1$.

6. 10 identical balloons will be distributed to 4 children. At least two balloons to each child. In how many many ways can this be done?

Answers or hints

1. There must be 1 or 2 digits. For one digit we have $40 = 10 \times 4$ possibilities, a seven in position 3 is one of them. Say we take two upper- and one lower case. $\binom{26}{2}$ possibilities for the upper case and 26 for the lower. KLb is such a choice and these three can be permutated in six ways on the remaining three positions. All in all, for only one digit we have $40 \times \binom{26}{2} \times 26 \times 6 \times 2 = 4.056 \cdot 10^6$ passwords. The last factor 2 appears since we also have to take the case with two lower case letters in account. Kb7L is such a password. If you use two digits you have $\binom{10}{2} \times \binom{4}{2} \times 2$ possibilities. For example 3 and 8 in position 2 and 3. Say you take X and p for the letters then you can form X38p and p38X. You have $26 \times 26 \times 2$ ways to put letters in position 1 and 4. So for two digits we get $\binom{10}{2} \times \binom{4}{2} \times 2 \times 26 \times 26 \times 2$ passwords. Finally you add the two numbers to get the total number of passwords.

2. $\frac{8!}{2!2!2!} = 5040$. The answer to the harder question is 960. If K is in position 2 we must put an R first and then we have 6 possibilities for the second R. Say you put it in position 6. Then you have $3 \times 2 = 6$ possibilities for the two B's. For example you can put them in position 4 and 7. Then you have 3 choices for Q and then you are forced to place H and H on the remaining positions. In total, $6 \times 3 \times 2 \times 3 = 108$. RKQBHRBH is such a word. Then you continue with K in position 3 and 4. Due to symmetry we get also 108 when K is in position 7 et cetera.

   Yes this is about chess! This old game has nowadays got a new twist, it is called chess960 and you can read about it and play it on the internet. The standard position is one of these 960. R stands for rook, H means horse or knight, B bishop, Q queen and K king.

3. $\binom{10}{8} = \binom{10}{2} = 45$. 8 sticks and 2 plus signs.

4. $\binom{15}{13} - \binom{3}{1}\binom{9}{7} + \binom{3}{2}\binom{3}{1}$.

5. This is about the simplest one-dimensional cellular automata in the project.

   (a) $2^3 = 8$

   (b) $2^8 = 256$

   (c) 2, all goes to 0 or all goes to 1.

   (d) $256 - 3 \cdot 2^4 + 3 \cdot 2^2 - 2 = 218$.

6. When you have given each child 2 balloons you have only 2 left. These you can distribute in $\binom{5}{2} = 10$ ways.