Linnaeus University
Mathematics
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Problems counting techniques II

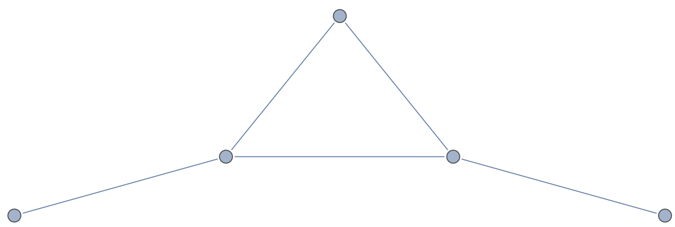
1. Find the generating function, $G(x)$ for the following problem: In how many ways can 20 identical ballons be distributed to 4 children so that each child gets at least 3 ballons but no one gets more than 7 ballons. Express $G(x)$ on closed form. Explain how to proceed to solve the problem.

2. Some short combinatorial questions.

   a) What is the sum of the n:th row in Pascal's triangle?
   b) How many six letter strings can be formed by 2 U, 2 R and 2 D?
   c) How many non-negative integer solutions are there to the equation

   $$x_1 + x_2 + x_3 = 13 \quad ?$$

   d) Give on closed form the generating function for an infinite string of ones, that is 1, 1, 1, 1, ...
   e) In how may ways can you make a proper coloring of the graph below if you have 5 colors available? In such colorings vertices with an edge in common must have different colors.



3. Give the generating function for the sequence $1, 0, 1, 1, 1, 1, 1 \ldots.$

4. This problem is from last week but this time you solve it with generating function. How many integer solutions are there to the problem

   $$x_1 + x_2 + x_3 = 28 \quad ?$$

   if $4 \le x_1 \le 9, \quad 5 \le x_2 \le 10, \quad 6 \le x_3 \le 11$?

5. Determine the number of integer solutions to the equation

   $$x_1 + x_2 + x_3 + x_4 + x_5 = 20$$

   when $1 \le x_i \le 6$, $i = 1, 2, 3, 4, 5$. That is, find the number of ways to get the sum 20 when throwing five dice. Use the generating function.

Answers or hints

1. First we put the generating function on compact form.

$$G(x) = (x^3 + x^4 + x^5 + x^6 + x^7)^4 = x^{12}(1 + x + x^2 + x^3 + x^4)^4 = x^{12}(\frac{1 - x^5}{1 - x})^4$$

Then we unzip it. That is, we use binomial theorem and table to expand the expressions.

$$G(x) = x^{12}(1 - 4x^5 + 6x^{10} \cdots + x^{20})(1 + \binom{4}{1}x + \binom{5}{2}x^2 + \binom{6}{3}x^3 \cdots + \binom{11}{8}x^8 \cdots)$$

Now it is time to determine the coefficient in front of x^{20}. It is $\binom{11}{8} - 4\binom{6}{3}$.

2. (a) $2^n = (1 + 1)^n$.
   (b) $\frac{6!}{2!\cdot 2!\cdot 2!} = \binom{6}{2}\binom{4}{2}\binom{2}{2} = 90$.
   (c) $\binom{15}{13} = \binom{15}{2}$. 13 sticks and 2 plus signs.
   (d) $\frac{1}{1-x}$. The geometric series.
   (e) $5 \cdot 4^3 \cdot 3 = 960$.

3. $1 + x^2 + x^3 + \cdots = 1 + x^2(\frac{1}{1-x}) = \frac{1-x+x^2}{1-x}$.

4. First we put the generating function on compact form.

$$G(x) = (x^4 \cdots + x^9)(x^5 \cdots + x^{10})(x^6 \cdots + x^{11}) = x^{15}(1 + x \cdots + x^5)^3 = x^{15}(\frac{1 - x^6}{1 - x})^3$$

Then we use table and binomial theorem so we can expand $G(x)$ again.

$$G(x) = x^{15}(1 - 3x^6 + 3x^{12} - x^{18})(1 + \binom{3}{1}x + \binom{4}{2}x^2 \cdots + \binom{9}{7}x^7 \cdots + \binom{15}{13}x^{13} + \cdots)$$

The coefficient in front of x^{28} is $\binom{15}{13} - 3\binom{9}{7} + 3\binom{3}{1}$.

5. Same procedure as in previous problems.

$$G(x) = (x + x^2 + \cdots + x^6)^5 = x^5(1 + x + x^2 + x^3 + x^4 + x^5)^5 = x^5(\frac{1 - x^6}{1 - x})^5$$

$$G(x) = x^5(1 - 5x^6 + 10x^{12} - 10x^{18} + 5x^{24} - x^{30})(1 \cdots + \binom{7}{3}x^3 \cdots + \binom{13}{9}x^9 \cdots + \binom{19}{15}x^{15} \cdots)$$

The coefficient in front of x^{20} is $\binom{19}{15} - 5\binom{13}{9} + 10\binom{7}{3} = 651$. Sum 20 occurs in $\frac{651}{6^5} \approx 8.4$ percent of the cases.