Linnaeus University
Mathematics
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Problems relations

1. Consider the set $A = \{1, 2, 3, 4\}$. On the cartesian product $A \times A$ we define the relation $\mathcal{R}$ by

$$(x_1, y_1)\mathcal{R}(x_2, y_2) \iff y_1 - x_1 = y_2 - x_2$$

   Show that $\mathcal{R}$ is an equivalence relation and illustrate the different equivalence classes in a figure. Note this is a relation on a set of points in the plane

2. Consider the following set

$$S = \{(a, b) \mid a, b \in \mathbf{Z}, b \neq 0\}$$

   where $\mathbf{Z}$ denotes the integers. Show that the relation

$$(a, b)\mathcal{R}(c, d) \iff ad = bc$$

   on S is an equivalence relation. Give the equivalence class $[(1, 2)]$. What can an equivalence class be associated with?

3. Two Hasse diagrams are shown below. Unfortunately the labels at the vertices are missing. Give suggestions of what they can be. For the first diagram the relation is the divides relation, that is $x\mathcal{R}y$ if and only if $x|y$, and the set is some subset of the positive integers. For the last diagram the relation is the inclusion relation, that is $x\mathcal{R}y$ if and only if $x \subseteq y$, and the set is a subset of the power set for a finite set.

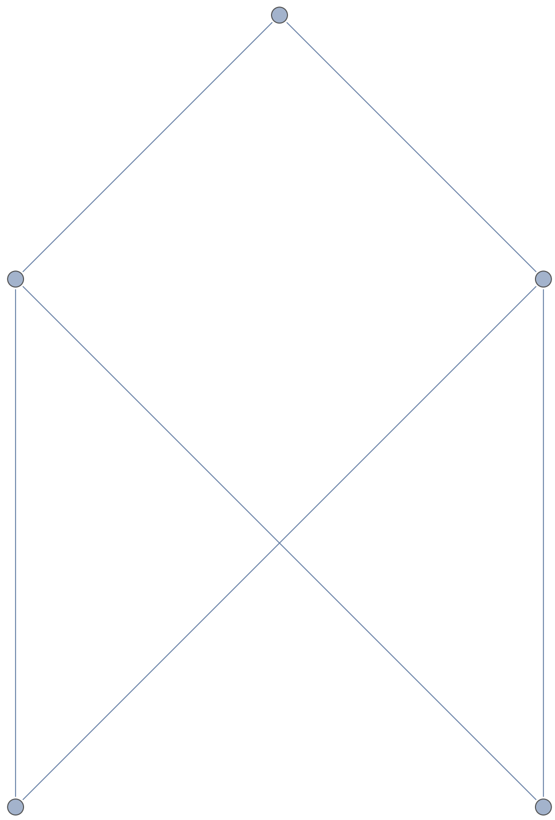
4. Draw the Hasse diagram for divisibility on the set $\{1,\ 2,\ 3,\ 5,\ 10,\ 15,\ 30,\ 45,\ 90\}$. Is it a lattice?

5. a) Determine whether the relation represented by this zero-one matrix is an equivalence relation.

$$\mathrm{M}_R = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{pmatrix}$$

   b) Determine whether the relation represented by the directed graph in figure 3 is an equivalence relation.

6. a) Show that the divides relation is a partial order on the positive integers. The divides relation, $\mathcal{R}$, is defined by: $x\mathcal{R}y$ if and only if $x|y$. Here x and y are positive integers.
   b) Define the relation $\mathcal{R}$ on the the integers $\mathbb{Z}$ by $x\mathcal{R}y$, for $x, y \in \mathbb{Z}$, if and only if $x \equiv y \pmod 7$. Show that this is an equivalence relation on $\mathbb{Z}$.



Figur 1: First diagram for problem 3

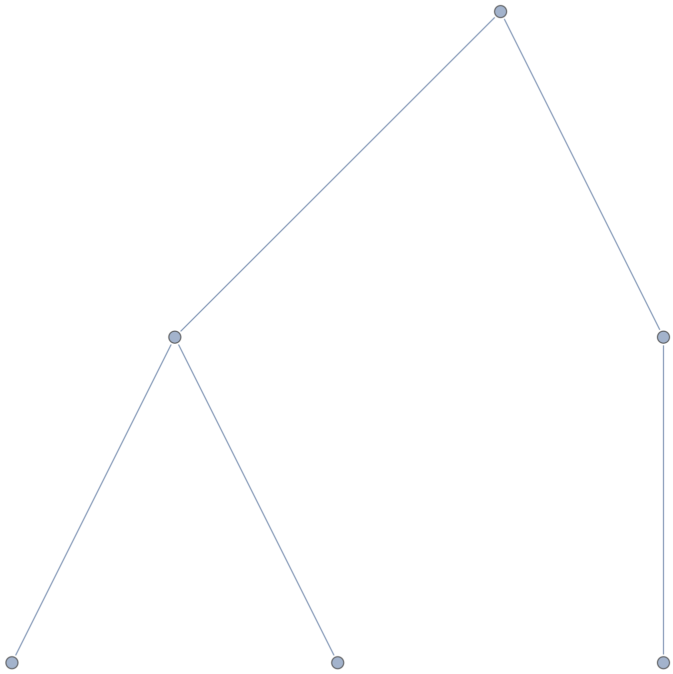
7. Some short questions on relations:
   a) Give an example of a Hasse diagram for a *totally ordered set*.
   b) Give a *partition* of the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ formed by three subsets A_1, A_2, A_3. Moreover, $|A_1| = |A_2| = |A_3|$.
   c) Draw a *digraph* for a reflexive and transitive binary relation, R, on a set $A = \{a, b, c\}$ where $(a, b) \in R$ and $(a, c) \in R$. $(a, b) \in R$ means that a is related to b.

8. Two short questions on relations and combinatorics:

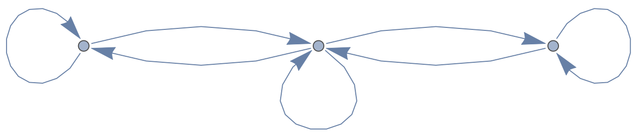
   (a) How many relations are there on set a V with three elements, $V = \{x, y, z\}$?

   (b) How many equivalence relations are there on set a V with three elements, $V = \{x, y, z\}$? Think of the number of partitions!

9. Let $A = \{1, 2, 3, 4, 6, 9, 12, 18, 36\}$, the set of positive divisors to 36. Define the relation R on A by $aRb \Leftrightarrow a|b$ (the divides relation). Draw the Hasse diagram for this partial order. Find the least upper bound of $\{2, 9\}$.



Figur 2: Second diagram for problem 3



Figur 3: Directed graph in problem 5b

Answers or hints

1. Reflexive? Yes since $y_1 - x_1 = y_1 - x_1$. Symmetric? Yes since $y_1 - x_1 = y_2 - x_2 \to y_2 - x_2 = y_1 - x_1$. Transitive? Yes since $y_1 - x_1 = y_2 - x_2$ and $y_2 - x_2 = y_3 - x_3 \to y_1 - x_1 = y_3 - x_3$. The members of an equivalence class lie on a straight line. For example $[(1,2)] = \{\cdots (0,1), (1,2), (2,3) \cdots\}$.

2. These are the rational numbers, one equivalence class is $\frac{1}{2} = \frac{2}{4} = \frac{3}{6} \cdots$. We see it clear if we write the equality as $\frac{a}{b} = \frac{c}{d}$. Now I think you can show that the relation is reflexive, symmetric and transitive. For transitivity you have to introduce a third point (e, f).

3. Figure 1 from left to right and down to up: 2,5,30,70,210 is one example.
   Figure 2 from left to right and down to up: $\{1\}, \{2\}, \{3\}, \{1,2\}, \{3,4\}, \{1,2,3,4\}$ is one example. These subsets are subsets of the power set with four elements.

4. See figure 4. Yes it is a lattice.

5. a) Yes. The matrix is symmetric and there are ones along the diagonal. For transitivity we can observe we have two blocks with ones everywhere. $1, 2, 5$ is one equivalence class and $3, 4$ the other. If we interchange the labels of 3 and 5 this block structure will be evident.

b) No. Left element is not related to right element but left to middle and middle to right are so transitivity fails.

6. I only show transitivity here and leave to you to show that the relation is also reflexive and anti-symmetric (a) or symmetric (b).
   a) $x|y$ means $y = kx$ for some integer k. $y|z$ means $z = my$ for some integer m. So combine these equalities and we get $z = my = mkx \to x|z$ and therefore $x\mathcal{R}z$.

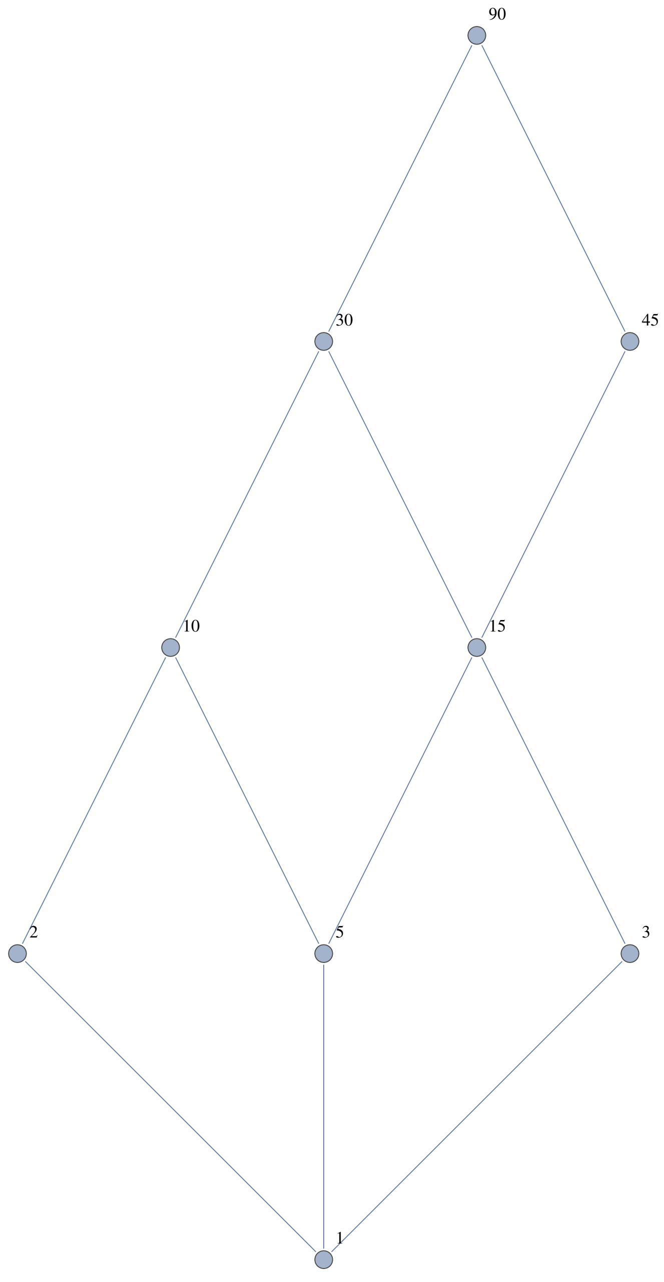
   b) $x\mathcal{R}y$ means $y - x = k7$ for some integer k. $y\mathcal{R}z$ means $z - y = m7$ for some integer m. So add these equalities and we get $z - x = (k + m)7$ and therefore $x\mathcal{R}z$.

7. a) The tree is just a trunk. Like the divides relation on 1,2,4,8.
   b) Join them 4 by 4. $48 = 3 \times 16$ elements in this relation on A.
   c) See figure 5.

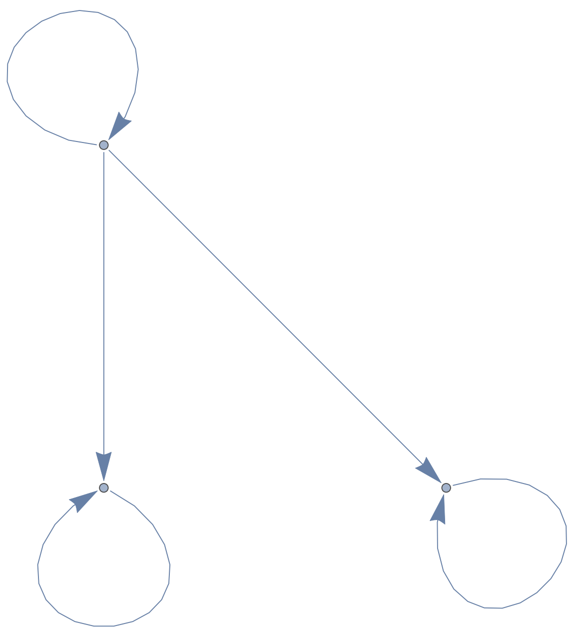
8. a) $2^9 = 512$
   b) Five. All together is one, all alone is another. Then we have three of the type 2+1.

9. See figure 6. $lub\{2, 9\} = 18$

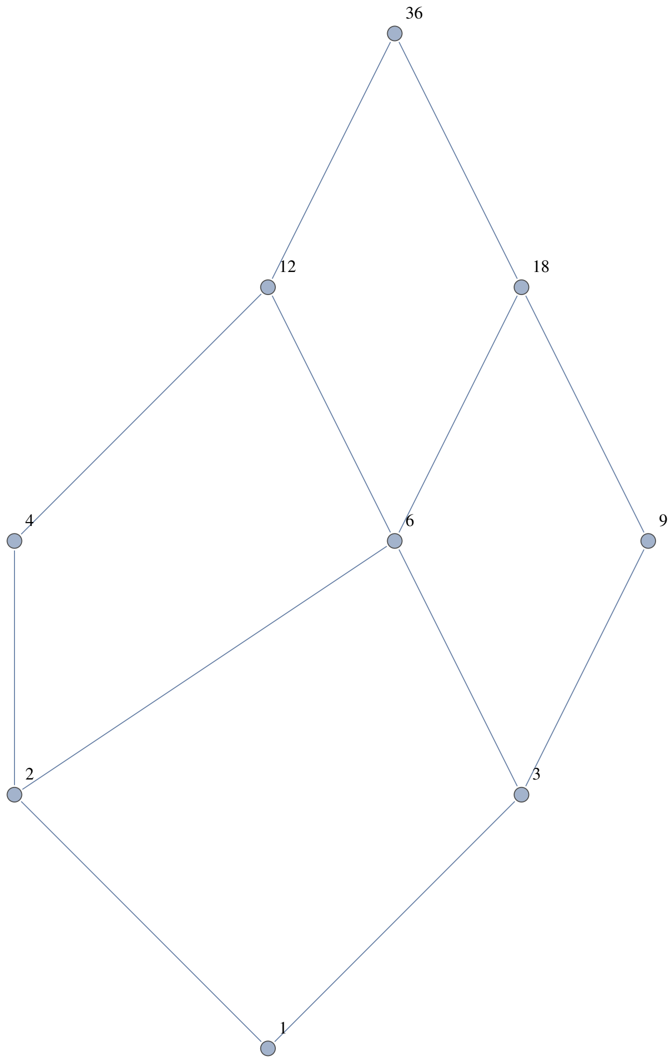
Out[3]=



Figur 4: Hasse diagram for problem 4



Figur 5: Directed graph for problem 7c



Figur 6: Hasse diagram for problem 9