Linnaeus University
Mathematics
Hans Frisk

Problems graph theory I

1. Draw the following graph. Each vertex corresponds to a bit string of length 3 (there are eigth of them). There is an edge between two vertices if and only if the two bit strings differ in exactly one position. So for example, there will be an edge between 101 and 100.

2. Consider the graphs Q_n, $n = 1, 2, 3, \cdots$. You draw Q_3 in previous problems. How many vertices and edges in Q_n? Think of bit strings of length n. Remember, there is an edge between two vertices if and only if the two bit strings differ in exactly one position. Use the handshake theorem.

3. For which positive integers n is C_n (the cycles of length n) bipartite?

4. Draw a graph G with 5 vertices such that G is isomorphic to $\overline{G}$. The complementary graph $\overline{G}$ has the same number of vertices as G but two vertices in $\overline{G}$ are ajacent if and only if they are not adjacent in G

5. Investigate simple connected graphs, G, with 15 vertices such that five of them have degree 4 and the remaining 10 vertices have degree 1.
   (a) How many edges in G?
   (b) Is G planar? If so, how many regions? A graph is planar if we can draw it without edge crossings. Then we get regions (triangles, squares, pentagones et cetera) and Euler's formula tells us that $r = e - v + 2$ where r denotes the number of regions.
   (c) Are there non-isomorphic graphs of this type?
   (d) Finally you make drawings of the possible graphs.

6. The *odd graph* O_k, k is an integer ≥ 2, is defined in the following way: The vertices represent the subsets with $k-1$ elements that can be obtained from a set with $2k-1$ elements. Two vertices are joined with an edge if and only if the corresponding subsets are disjoint.
   (a) Draw O_2 and O_3.
   (b) Show that O_k is k-regular for all $k \geq 2$, that is, show that all vertices in O_k has degree k.

Answers or hints.

1. See exam question 180524-3a in old exams folder.

2. Q_n has 2^n vertices and each of them has degree n. Using handshake theorem we get $2e = n2^n$ so $e = n2^{n-1}$. For example the ordinary cube (n=3) has 12 edges.

3. n must be an even integer. Then we can use two colors to do a proper coloring. A graph is is bipartite $\Leftrightarrow$ The chromatic number is 2.

4. For example the pentagon and the pentagram in K_5.

5. (a) $2e = 5 * 4 + 10 * 1 = 30$ so e=15. The graphs are like trees but with one extra edge added. If we think of chemistry this is molecule C_5H_{10} and there are some isomers.
   (b) $r = 15 - 15 + 2 = 2$. Yes these graphs are planar. One cycle and one outer region.
   (c) Yes I think they are five. $2e = 5 + 25 = 4 + 26 = 3 + 27$ (three possibilities for the last one but I am not sure if they all exist as molecules). There can only be degree 4 vertices in the inner cycle so it must have length 3,4 or 5.

6. See also exam question 180824-6 in old exams folder. O_2 is isomorphic to C_3. O_3 is isomorphic to the Petersen graph (see my notes for a picture). When we have picked a subset the number of remaining elements in O_k are equal to $2k - 1 - (k - 1) = k$. From them we can form $C(k, k - 1) = k$ subsets with $k - 1$ elements. Each of them are adjacent to the one we picked so the degree is k for all vertices in O_k.