Linnaeus University
Mathematics
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Problems graph theory II

1. The minimum number of colors needed for a proper coloring of a graph G is called the chromatic number and is denoted $\chi(G)$. Give the following chromatic numbers.
   a) $\chi(K_{12})$
   b) $\chi(K_{15,30})$
   c) $\chi(C_{17})$

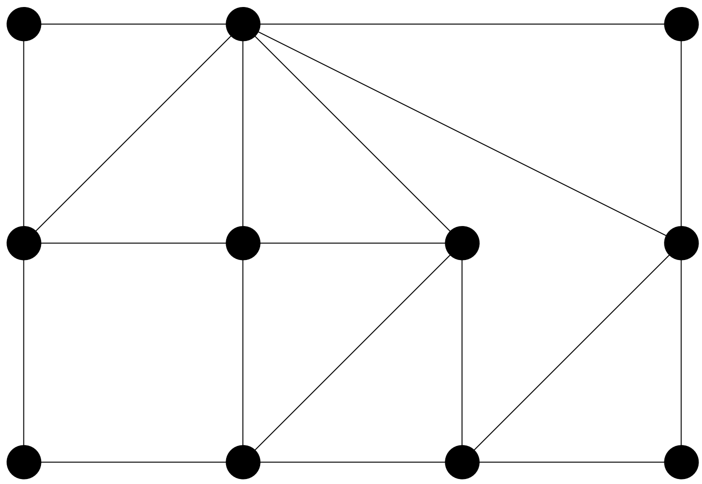
2. You want to color the vertices in C_3 and you have **three** colors available, red, blue and green.

   (a) How many colorings can be done?

   (b) How many of them are proper colorings?

3. For which m, n has $K_{m,n}$ a Hamilton cycle (HC). A HC goes over all vertices exactly once. Note, you don't have to pass over all edges. Think of a travelling salesman. The vertices are cities and the edges roads.

4. Is there an Euler circuit in the graph in figure 1? If so, try to find it. Find also a Hamilton



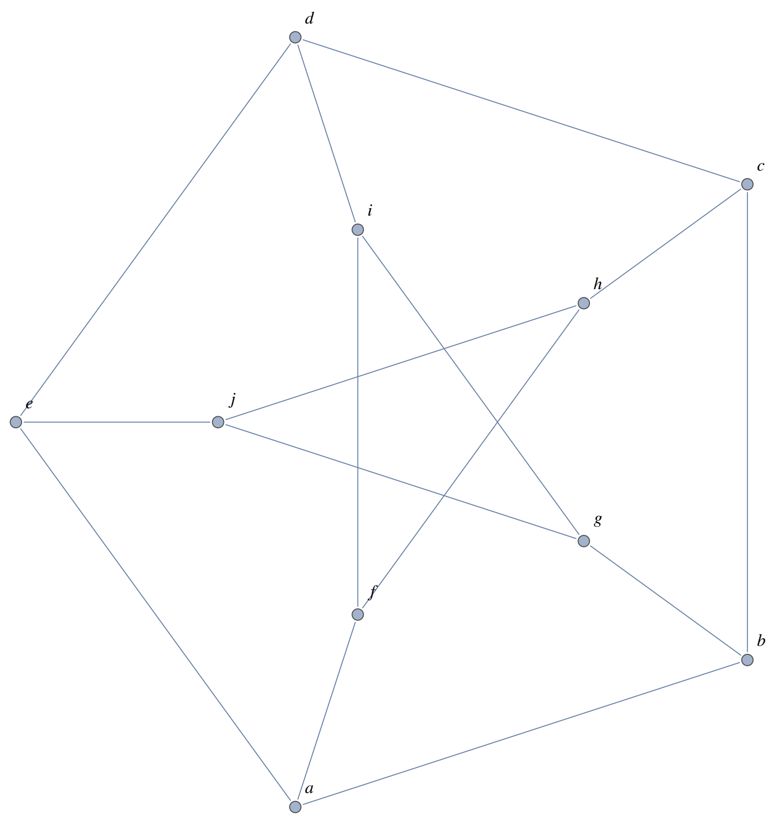
Figur 1: Find EC and HC

   cycle.

5. Draw a planar graph with $v = 6$ and $deg(v_i) = 4$ for all vertices $v_i, i = 1, 2 \cdots, 6$.

6. Below in figure 2 you find the Petersen graph, P. Show that $P - c$ is homeomorphic to $K_{3,3}$. $P - c$ is the subgraph of P where vertex c and all edges attached to it are deleted.

7. Let G be a simple graph with $|V| = n \geq 3$. If $\deg(x) + \deg(y) \geq n$ for all nonadjacent $x, y \in V$, then G contains a Hamilton cycle. This is a sufficient condition for the existence of a Hamilton cycle proved by Ore 1960.



Figur 2: The Petersen graph

(a) Draw a graph for which the condition holds and find the Hamilton cycle. Show also that it is not a necessary condition, that is, find a graph with a Hamilton cycle but for which the condition is not fulfilled.

(b) A group of 12 people meet for dinner at a big circular table. In this group everyone knows at least 6 others. Can they be seated around the table in such a way that each person knows the person to the left and to the right?

8. (a) Consider connected simple graphs, G, with 11 vertices. Prove that either G or its complement $\overline{G}$ must be nonplanar. For definition of complementary graph see problem 4 from last week

   (b) This theorem does not hold for eight vertices. Find a counterexample to part (a) above, that is, find a planar G for which also the complement is planar.

9. Show that $e \leq 3v - 6$ is a necessary condition for planar graphs. Use the Euler formula and assume that the regions are made of triangels, squares .... Show that in a planar graph not all vertices can have degree 6 or more. Use handshake theorem and inequality above.

10. The *girth* of a graph G is the length, i.e. the number of edges, of the shortest cycle it is possible to find in G and it is denoted $g(G)$. For graphs with n vertices and $n+1$ edges, $n \geq 4$, it is possible to show that

$$g(G) \leq \left\lfloor \frac{2(n+1)}{3} \right\rfloor$$

The floor function $\lfloor x \rfloor$ gives the largest integer $\leq x$.
a) Draw a graph with maximal girth for $n = 5$.

b) To show the inequality above one starts by first showing that a graph with n vertices and $n+1$ edges always has at least two cycles. Do that!

Answers or hints.

1. See exam question 190524-2 in old exams folder.
2. (a) $3^3 = 27$
   (b) $3 \cdot 2 \cdot 1 = 6$
3. See exam question 190611-2f in old exams folder.
4. See exam question 190824-1 in old exams folder.
5. $e = 12$. $r = 12 - 6 + 2 = 8$. This is the octahedron. Draw C_6 and order the vertices 1 to 6 counter-clockwise. Connect 1,3 and 5 inside the cycle and 2,4 and 6 outside the cycle.
6. I did this in my recording. The subgraph $G - \{c\}$ is homeomorphic to $K_{3,3}$. The vertices b, d, h are the elementary subdivisions.
7. (a) In C_4 the condition is fulfilled and the cycle is a HC. In the dodecahedron I showed in my lecture there $3 + 3 < 20$ but is not hard to find a HC.
   (b) Yes. $6 + 6 \geq 12$ so there is a HC. Use it for the ordering at the table.
8. (a) $e \leq 3v - 6$ is a necessary condition for planarity so 27 edges is the maximum number of edges if $v = 11$. K_{11} has 55 edges and $55 = 27{+}28 = 28{+}27 = 26{+}29 = 29{+}26 \cdots$ so one of the two graphs has too many edges.
   (b) As mentioned above $e \leq 3v - 6$ is a necessary condition for planarity so 18 edges is the maximum number of edges if $v = 8$. K_8 has 28 edges and $28 = 10{+}18 = 11{+}17 = 12{+}16 = 13{+}15 = 14{+}14$ so both graphs can be planar. I did it for 10+18 but it was tiresome. 14+14 is simpler I hope. It is isomorphic to the complement. $v = 8, e = 14$ so $r = 8$ and 4 vertices have degree 4 and 4 have degree 3. The same for the regions. Draw then Q_3 and add 2 more edges. You can draw a diagonal in the inner square and connect opposite vertices in the outer square outside Q_3.
9. See my lectures about graphs.
10. See also exam question 190611-5 in old exams folder. a) Draw a square and join the fifth vertex with two diagonal opposite vertices.
    b) A tree has no cycles and $e = n - 1$ edges. You have to draw two more edges. For each edge you create at least one cycle.