Linnaeus University
Mathematics
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Problems graph theory III

1. How many edges in a full binary tree with 1000 internal vertices?

2. Which complete bipartite graphs, $K_{m,n}$ are trees?

3. Five short questions on graphs and trees:
   a) What is the chromatic number for $K_{4,3}$?
   b) Draw a full, rooted, binary tree with 8 leaves.
   c) Find a Hamilton cycle in K_5.
   d) For which n have Q_n an Euler circuit?
   e) Draw the non-isomorphic simple graphs with 4 vertices and 2 edges.

4. Draw all non-isomorphic trees with five vertices. Note trees are simple connected graphs.

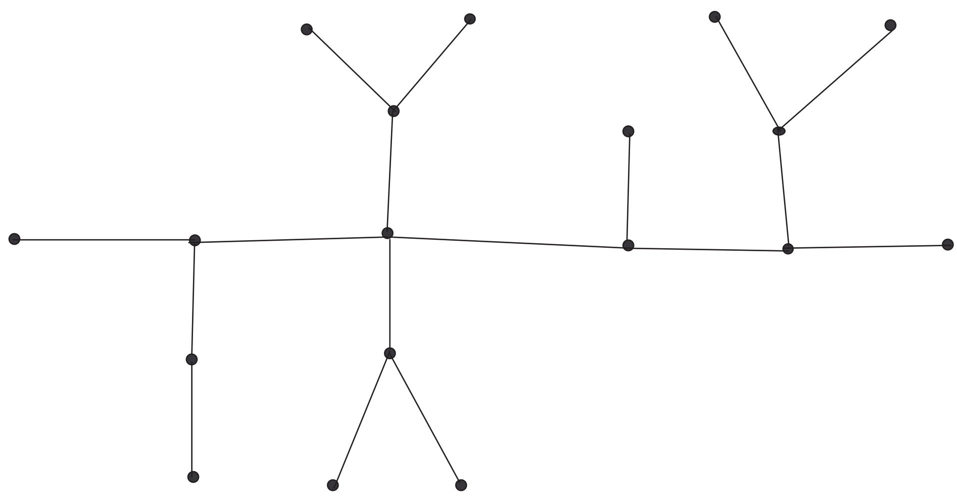
5. A tree in which each non-leaf vertex has a constant number of branches n, that is the same degree n, is called a n-Cayley tree. Below you are asked to draw some of them. Draw only non-isomorphic trees. Support your findings with mathematics!

   (a) Draw the 2-Cayley tree(s) with 4 vertices.

   (b) Draw the 5-Cayley tree(s) with 6 vertices.

   (c) Draw the 3-Cayley tree(s) with 10 vertices.

6. The *eccentricity* of a vertex in an unrooted tree is the length of the longest *simple path* beginning at this vertex. A vertex is called a *center* if no other vertex in the tree has smaller eccentricity than this vetex. Find every vertex that is a center in the tree below. A path is simple if it does not contain the same edge more than once.

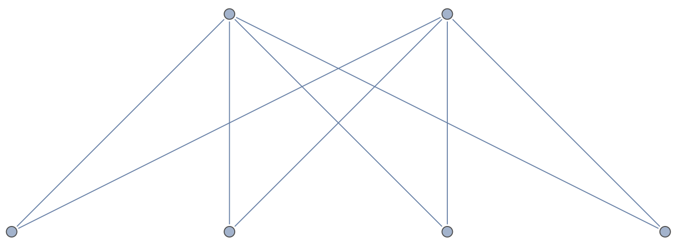


7. Why exactly five Platonic solids (PS)? A PS is a planar graph where all regions have the same degree m and all verices have the same degree n. For example the tetrahedron has $m = n = 3$. So $2e = mr = nv$. Using Euler's formula we get

$$0 < 2 = r - e + v = \frac{2e}{m} - e + \frac{2e}{n}.$$

For which m, n is this inequality fulfilled?

8. The complete bipartite graphs are denoted $K_{m,n}$ where m and n are positive integers. For illustration $K_{2,4}$ is shown below.



(a) Find a Hamilton cycle in $K_{3,3}$.

(b) How many edges has$K_{30,35}$?

(c) How many proper colorings of $K_{3,3}$ are there if 4 colors are available? Hint: here you have to use the sum rule. Sum over 3,2 and 1 color on the upper vertices.

9. Consider simple planar graphs with 6 vertices and 7 edges. How many regions do they have? Determine the possible degrees of the regions. Draw three non-isomorphic graphs of this type.

10. Ten short questions from the whole course

a) Give the number of non-negative integer solutions to the equation $x_1+x_2+x_3+x_4 = 14$.

b) Give a Hasse diagram for a *totally ordered set* (TOS) with five elements. For every pair a and b in a TOS we have aRb or bRa.

c) Show in a Venn diagram the set $A \cap \overline{B}$. A and B are two different non-empty sets with an intersection. $\overline{B}$ denotes the complement to set B.

d) Give the generating function for the sequence $1, 5, 10, 10, 5, 1, 0, 0, 0, 0, 0 \cdots$.

e) In which complete graphs, K_n $(n \geq 3)$, can we find an Euler circuit?

f) In which complete bipartite graphs can we find a Hamilton cycle?

g) How many proper colorings can be made of the vertices in C_3 if 5 colors are available?

h) How many bit strings of length seven are there?

i) Draw a tree with 3 internal vertices and 2 leaves.

j) How many functions are there from the set of bit strings of length 3 to the set of bit strings of length 1?

11. Seven short questions so the answers can also be short.

(a) What is the chromatic number for a tree?

(b) Which is the generating function for the finite sequence 1,4,6,4,1?

(c) V is a poset with six elements. Draw a possible Hasse diagram if the poset is a lattice.

(d) How many reflexive relations are there on the set $V = \{a, b, c, d, e\}$?

(e) Draw two non-isomorphic simple graphs with 5 vertices and 4 edges.

(f) How many words (letter combinations) can be formed by the letters in the word *rotator*?

(g) How many non-negative integer solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 = 12 \ \ ?$$

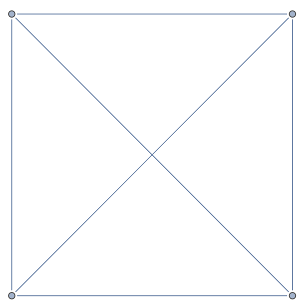
12. An interesting type of graphs is the *fullerene graphs*. They are planar and every vertex has degree 3. The regions are pentagons and hexagons. With pentagons and hexagons we mean C_5 and C_6 cycles, respectively.

(a) If a fullerene, G, has 60 vertices, how many edges and regions are there in G? How many of the regions are pentagons and how many are hexagons? (2p)

(b) Let p denote the number of pentagons and h the number of hexagons. Which values of p and h are possible in a fullerene graph? (2p)

(c) Draw the fullerene graph with 24 vertices. The drawing must be done without edge crossings since the graph is planar. (2p)

13. A spanning tree T to a graph G is a connected subgraph of G which is a tree and contains all the vertices in G. Let now $G = K_4$, see figure below.



a) Draw two non-isomorphic spanning trees in K_4.

b) Let us now also include isomorphic trees. How many spanning trees are there then in total in K_4?

Answers or hints.

1. Then 1001 leaves and 2001 vertices so there are 2000 edges.

2. $e = mn = v - 1 = m + n - 1$ so for which integers is $mn = m + n - 1$? If one or both are 1.

3. (a) 2 as for all bipartite graphs.
   (b) 7 internal vertices and all of them get 2 children. $v = 8 + 7 = 15$. KaryTree$[15, 2]$ is the Mathematica command to plot it.
   (c) Take the pentagon or the pentagram for example.
   (d) Each vertex in Q_n has degree n and this number must be even.
   (e) These graphs are not connected. They have two components. 4=2+2=3+1. In the last case 3 are connected and one disconnected vertex.

4. $e = 4$ since they are trees. Use handshake thorem. In how many ways can we write $2e = 8$ using five digits? The degrees lie between 1 and 4. There are 3 non-isomorphic trees. I give the degree sequence for them here $\{1, 1, 2, 2, 2\}, \{1, 1, 1, 2, 3\}, \{1, 1, 1, 1, 4\}$. The last one is called a star.

5. Handshake theorem again. $2e = 2(v - 1) = 2(L + I - 1) = nI + L$. This gives the following relation between the number of leaves L and the number of internal vertices I: $L - 2 = (n - 2)I$. Remember also that $v = L + I$.
   (a) $L = I = 2$.
   (b) $I = 1, L = 5$. This is a star, $K_{1,5}$.
   (c) $I = 4, L = 6, v = 10, e = 9$. Here opens up the possibility for non-isomorphic trees. I see two possibilities, the root get 6 grandchildren or the internals lie along a path.

6. The two vertices in the middle of the middle row have eccentricity 4. They are centers.

7. See my notes.

8. (a) Since $n = m$ it is possible (easy) to find a HC.
   (b) $30 \cdot 35 = 1050$ edges.
   (c) 3 colors on the upper vertices: $C(4, 3) \cdot 6 = 24$ ways to do a PC.
   2 colors on the upper vertices: $C(4, 2) \cdot 3 \cdot 2 \cdot 2^3 = 288$ ways to do a PC.
   1 color on the upper vertices: $C(4, 1) \cdot 3^3 = 108$ ways to do a PC.
   In total 420 proper colorings.

9. $r = 7 - 6 + 2 = 3$. It is like in a tree where you add two extra edges and create two new cycles. $2e = 14 = \deg R_1 + \deg R_2 + \deg R_3$. Remember $v = 6$ and the smallest degree for a region is 3. I found 5 different graphs.
   5+5+4. Take the square as the outer region and add two vertices on a diagonal and connect these vertices on the diagonal.
   6+4+4=6+5+3. Take the hexagon as the outer region and add long or short diagonal, respectively.
   7+4+3. Join a square with a triangle with one vertex in common.
   8+3+3. Take two triangles with one vertex in common and join a degree vertex 1 to this graph.

10. See exam question 190611-2 in old exams folder.

11. See exam question 180608-1 in old exams folder.
12. See exam question 180608-6 in old exams folder.
13. See exam question 190829-6 in old exams folder.