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Exam in Ordinary Differential Equations, 2MA101, 7,5 credit points
Friday 5th of November 2010, Time 8.00-13.00.

To obtain maximal point a complete solution, presented in such a way that calculations and reasoning are easy to follow, is demanded *Aid*: Dictionary

1. Consider the initial value problem

$$\mathbf{x}'(t) = \mathbf{f}(t, \mathbf{x}),$$
$$\mathbf{x}(t_0) = \mathbf{x}_0.$$

a) Reformulate the problem as an integral equation. (1p)
b) Describe Picard's method (successive approximations) and give a criteria for $\mathbf{f}(t, \mathbf{x})$ under which the method works. (2p)
c) Do the first two iterations with Picard's method for the initial value problem

$$\begin{aligned} x'(t) &= 1 + x^2 \\ x(0) &= 0 \end{aligned}$$

(2p)

2. Consider the undamped pendulum of length L

$$x''(t) = -\omega^2 sin\ x.$$

where the constant $\omega^2 = g/L$ (g is the gravitational constant)
a) Rewrite the equation as a system of first-order ODE's. (1p)
b) Where are the critical points located? (1p)
c) Is it an almost linear system? (1p)
d) Construct a Liapunov function and sketch the phase portrait. (2p)

3. Solve the separable equation

$$y'(x) = \frac{\sqrt{2 - y^2}}{y}.$$

when $y(0) = 1$ and sketch the solution in the upper half-plane. (5p)

4. Find the critical points in $the\ first\ quadrant$ to the system

$$\begin{aligned} x'(t) &= 12 - xy \\ y'(t) &= x^2 + y^2 - 25 \end{aligned}$$

and determine their character and stability. (5p)

5. Determine the solution to the initial value problem

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{x} + te^t \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \mathbf{x}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

(6p)

6. Consider the system

$$\begin{aligned} x_1'(t) &= \omega_{12} x_2 \\ x_2'(t) &= -\omega_{12} x_1 \\ x_3'(t) &= \omega_{34} x_4 \\ x_4'(t) &= -\omega_{34} x_3 \end{aligned}$$

where ω_{12} and ω_{34} are positive real constants.
a) Find the solution with the initial value $\mathbf{x}(0) = (0, 1, 1, 0)$. (3p)
b) Introduce the *periods* $T_{12} = \frac{2\pi}{\omega_{12}}$ and $T_{34} = \frac{2\pi}{\omega_{34}}$. If all solutions to the system are periodic orbits, i.e. there is a time τ such that $\mathbf{x}(t + \tau) = \mathbf{x}(t)$, what is then the relation between T_{12} and T_{34}? (4p)

7. A set D in $\mathbb{R}^2$ such that $\mathbf{x}(0) \in D \Rightarrow \mathbf{x}(t) \to \mathbf{0}$ when $t \to \infty$ is called the region of asymptotic stability for $\mathbf{0}$. Consider the system

$$\begin{aligned} x' &= -x - y^2, \\ y' &= -y - x^2. \end{aligned}$$

Here $\mathbf{0}$ is an asymptotic stable critical point. Find, for example by using a suitable Lyapunov function, the largest possible $r > 0$ such that the *open* circular disc with center at the origin and radius r lies in the region of asymptotic stability for $\mathbf{0}$. (7p)

LYCKA TILL! GOOD LUCK!