Mathematics
DFM
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Ordinary differential equations
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Introduction to ODEs

The simplest ODE you could think of is the **first order** equation

$$x'(t) = v(x), \, x \in R$$

where $v(x)$ is a known function of x that we may consider as a velocity. The parameter t denotes time. To the equation above is included an initial value, $x(0) = x_0$. For this **initial value problem** we want to reconstruct the history, $x(t) \quad t < 0$, or predict the future, $x(t) \quad t > 0$. If we think of v as a velocity then it is close to think of x as a point particles position. The region where the particle can move, for example $\mathbb{R}$, is called **phase space.** We can note that the (vector) field $v(x)$ is independent of time. Such dynamical systems are called **autonomous.** We shall also consider some non-autonomous systems in this course, i.e. v depends explicitly on time, $v(x,t)$. Such systems appear when the system (a planet, the concentration of a chemical, a heart...) is disturbed from an external source (comet, chemical additives, electrical shocks,...). You should know that to many ODEs you can not write down the solution in explicit form, an example is when $v(x,t) = x^2 - t$, and then we have to rely on numerical or approximate solutions. Hence we often concentrate on a qualitative understanding of the motion. It is then particular important to know the behaviour near **stationary points** (or critical points), i.e. those points where there is no movement at all, $v(x) = 0$. However, let us now look at some problems where there is an explicit solution.

Example 1 $v(x) = -x$, $x(0) = -10$. *Sketch the vector field. Are there any stationary points? How long time does it take for the particle to reach the stationary point? Sketch some **solution curves** for some different initial points in the **extended phase space** (x,t).*

Example 2 $v(x) = 3x^{2/3}$, $x(0) = 0$. *Same questions as above. OBSERVE: this is a non-linear ODE.*

Example 3 $v(x) = x^2$, $x(0) = 1$. *Also non-linear. Does the solution exist for all times?*

From the last example above we learn that the solution may not **exist** for all times, a catastrophe can occur! Contrary to example 1 the particle goes off to infinity in finite time. Also for Newton's N-body gravitational problem it is shown that bodies can move out to infinity in a finite time (Saari and Xia, Notices of the AMS, Vol 42, p538). In example 2 we found a solution but it is not **unique!** In fact for every $\alpha \geq 0$ we can construct a solution in the following way

$$\begin{aligned} x_\alpha(t) &= (t+\alpha)^3, t < -\alpha \\ x_\alpha(t) &= 0, -\alpha \le t < \alpha \\ x_\alpha(t) &= (t-\alpha)^3, t \ge \alpha \end{aligned}$$

This is also an interesting situation. In the course we will see that the reason for this ambiguity in example 2 is that $v'(x)$ does not exist at the origin.

In the next example it is not easy to find an explicit solution but we can anyhow get a qualitative feeling for the motion.

Example 4 $v(x) = \sin x$. *Think about what will happen for different start-values* $x(0)$. *In fact it is possible to find an explicit solution by the substitution* $x = 2arctan\ u$. *If* $x(0) = \pi/2$ *the solution becomes* $x(t) = 2arctan(e^t)$.

Let us now go further with a 2-dimensional phase space. The ODE is now $\mathbf{x}' = \mathbf{v}$ where $\mathbf{x} = (x_1, x_2)$ and $\mathbf{v} = (v_1, v_2)$.

Example 5

$$\begin{aligned} x_1'(t) &= x_1 \\ x_2'(t) &= 2x_2 \end{aligned}$$

Here $\mathbf{v} = (x_1, 2x_2)$.*Observe that the equations are two 1D ODEs; the two equations do not "talk" to each other, they are not coupled.*

If we introduce the matrices

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

then the equation can be written in the form

$$\mathbf{x}' = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \mathbf{x}.$$

$(0,0)$ *is the only stationary point. If the initial point is not on a coordinate axis then* ***phase curves*** *are (do the calculations!)*

$$| x_2 | = C | x_1 |^2$$

where C *is determined by* $\boldsymbol{x}(0)$. *How do the phase-curves look like if you start on a coordinate axis? Draw some phase-curves and obtain from those a* ***phase portrait.***

Example 6 *Draw the phase portrait for* $\boldsymbol{v} = (x_1, 0)$ *and* $\boldsymbol{v} = (x_1, -x_2)$.

Now we do it somewhat more diffuclt by coupling the equations. We start with an example from discrete mathematics. The theory for discrete time is closely related to the continous case that we study in this course, but see also section 2.9 in the book.

Example 7 *Fibonaccis number sequence*

$$0,1,1,2,3,5,8,13,........$$

is defined by the initial conditions $x_0 = 0$, $x_1 = 1$ *and*

$$x_n = x_{n-1} + x_{n-2},\ n \geq 2$$

Can we find a function $F(n)$ *such that* $x_n = F(n)$ *? Yes, introduce the vector* $\mathbf{x}_n = (x_n, x_{n-1})$ *where* x_n *and* x_{n-1} *is included in the Fibonacci sequence. Start value is* $\mathbf{x}_1 = (1,0)$. *What is* $\mathbf{x}_2$ *? Put* $\mathbf{x}_n = A\mathbf{x}_{n-1}$. *How does the matrix* A *look like?* A *is not on diagonal form like in example 5 so we would like to find a coordinate transformation* T *from* $\mathbf{x}_n$ *to new coordinates* $\mathbf{y}_n$, $\mathbf{x}_n = T\mathbf{y}_n$, *such that the mapping is on diagonal form in the new coordinates*

$$\mathbf{y}_n = T^{-1}AT\mathbf{y}_{n-1} = D\mathbf{y}_{n-1}.$$

D *denotes the diagonal matrix. From linear algebra we know that the columns of the matrix* T *must then be the eigenvectors of* A. *The solution is simple to find in the new coordinate system,* $\mathbf{y}_n = D^n\mathbf{y}_1$. *Let the eigenvalues to* A *be denoted by* λ_1*och* λ_2, $\lambda_{1,2} = \frac{1\pm\sqrt{5}}{2}$, *then we get by help of* $\mathbf{x}_n = T\mathbf{y}_n$ *that (do the calculations!)*

$$x_n = \frac{1}{\sqrt{5}}\left(\lambda_1^n - \lambda_2^n\right)$$

We here saw that some linear algebra is needed when you study **linear systems** and that is also true in the continuous case to which we now return.

Example 8 *Sketch the phase portrait when* $\boldsymbol{v} = (x_2, x_1)$. *Hint: Do variable change* $y_1 = x_1 + x_2$,. $y_2 = x_1 - x_2$. *How does the system of differential equations look like in the new coordinates? How is A looking like? Determine the eigenvalues and eigenvectors and the coordinate transformation* T.

If you compare the linear equation $x'(t) = ax(t)$ that has the solution $x(t) = e^{at}x(0)$ with the linear system of differential equations $\mathbf{x}'(t) = A\mathbf{x}(t)$ then it is tempting to write the solution as $e^{At}\mathbf{x}(0)$. This turns out to be correct if we define the exponential function of a matrix as the matrix

$$e^{At} = I + At + A^2t^2/2! + A^3t^3/3! + ...$$

where I is the unit matrix. e^{At} is called the **fundamental matrix**.

Here follows some interesting examples from biology and physics.

Example 9 *An example from population dynamics is*

$$\begin{aligned} x_1'(t) &= x_2 + x_1\left(1 - x_1^2 - x_2^2\right) \\ x_2'(t) &= -x_1 + x_2\left(1 - x_1^2 - x_2^2\right) \end{aligned}$$

that is non-linear and those are mostly not solved explicitly. Understanding of the solutions can in this case be obtained by going to polar coordinates, $x_1 = r\cos\theta$ $x_2 = r\sin\theta$,

$$\begin{aligned} r' &= r\left(1 - r^2\right) \\ \theta' &= -1. \end{aligned}$$

, $r \geq 0$. *The first equation has two stationary points and* $r = 1$ *is a* ***limit cycle*** *going around the unit circle in clockwise sense.*

Example 10 *Newton's second law for a ball attached to a spring ($F = -kx = ma$ and mass and spring coefficient are put to 1) is*

$$x'' = -x$$

If we introduce $x_1 = x$ and $x_2 = x'$ then we get the following system

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -x_1 \end{aligned}$$

The advantage is that there is only first order derivatives, the disadvantage is that we get 2 equations but as we shall see in the course the advantages generally outweighs the disadvantages! How is the phase portrait to the oscillating spring looking like?
Characteristic for mechanical systems where the energy is constant is that there is a function $H(x_1, x_2)$ such that $x_1' = \partial H/\partial x_2$, $x_2' = -\partial H/\partial x_1$. What is H for the oscillating spring? What represent the 2 terms in $H(x_1, x_2)$?

Example 11 *Newton's 2nd law for an oscillating pendulum is with suitable coordinates*

$$x'' = -\sin x$$

where x is the angle which gives the location of the pendulum. Transform to a system and try to sketch a phase portrait. What happens for small angles x ?

Here is an example of an non-autonomous ODE.

Example 12 *Solve $x' = v(x,t) = \left(1 + e^t\right) x^2$, $x(0) = 1$ and compare with example 3 above. If we want we can also introduce coordinates $x_1 = x$ and $x_2 = t$ and then get an autonomous system*

$$\begin{aligned} x_1' &= (1 + e^{x_2}) \cdot x_1^2 \\ x_2' &= 1. \end{aligned}$$

The last 50 years an intensive research has dealt with so called chaotic systems. Chaos means that 2 different solutions to an ODE , with a bounded phase space, that start close to each other at time $t = 0$, diverge exponentially from each other by increasing time. Chaos can not exist in 1D and 2D phase space but appear first in a 3D phase space, e.g. a movement in the plane where $\mathbf{v}$ depends explicitly on time.

Finally I will give a fascinating example of what you can model with an ODE. There are so called chemical watches, e.g. the BZ-reaction (see http://taylor.mc.duke.edu/~rubin/BZ/BZexplain.html) where a chemical solution changes colour repeated times (not true with coffee and milk!). A model for BZ is

$$\begin{aligned} x' &= s(y - xy + x - qx^2) \\ y' &= \frac{1}{s}(-y - xy + fz) \\ z' &= w(x - z) \end{aligned}$$

s, q, f, w are parameters and $x = [HBrO_2]$, $y = [Br^-]$, $z = \left[Ce^{4+}\right]$, i.e. concentrations. There are two stationary points, $(0,0,0)$ trivial and $x_0 = z_0$, $y_0 = fx_0/(1 + x_0) =$

$(qx_0 - 1)\,x_0/(1 - x_0)$. The limit cycle (chemical clock) appears when the latter stationary point becomes unstable.