Systems of First Order Linear Equations

7.1

Subsystems "talk" to each other

Ex) $R(t) = \#$ rabbits prey

$F(t) = \#$ foxes predator

$$\frac{dR}{dt} = a_1 R - b_1 R \cdot F$$

$$\frac{dF}{dt} = -a_2 F + b_2 R \cdot F$$

$a_1, b_1, a_2, b_2 \in \mathbb{R}^+$

Lotka-Volterra
1925, 1926

a_1 - growth rate of prey
a_2 - death rate of predator

Note it is a non-linear system. See chapter 9. First we have to do the linear systems in chapter 7.

Ex 7.1.2) $u''(t) + \frac{u'(t)}{2} + 2u = 3\sin t$

Put $x_1(t) = u(t)$ and $x_2(t) = u'(t)$

$x_1'(t) = u'(t) = x_2$, $x_2' = u'' = \frac{-u'}{2} - 2u + 3\sin t$
$= \frac{-x_2}{2} - 2x_1 + 3\sin t$

$$\frac{d\bar{x}}{dt} = \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -2 & -\frac{1}{2} \end{pmatrix}}_{=A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ 3\sin t \end{pmatrix}}_{=\bar{g}}$$

$$\bar{x}' = A\bar{x} + \bar{g}$$

If $\bar{g} = \bar{0}$ for all times we have a homogeneous system. Otherwise it is a nonhomogeneous system.

Sometimes A is time-dependent

$$\bar{x}' = P(t)\bar{x} + \bar{g}$$

compact notation for

$$\begin{aligned} x_1(t) &= P_{11}(t)x_1 + \cdots + P_{1n}(t)x_n + g_1(t) \\ &\vdots \\ x_n(t) &= P_{n1}(t)x_1 + \cdots + P_{nn}(t)x_n + g_n(t) \end{aligned}$$

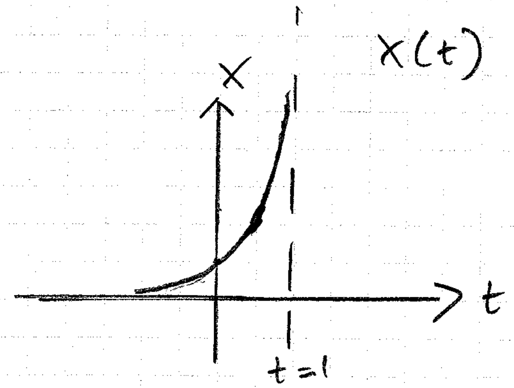
In non-linear case

$$\begin{aligned} x_1' &= F_1(t, x_1, x_2 \ldots x_n) \\ x_2' &= F_2(t, x_1, x_2 \ldots x_n) \\ &\vdots \\ x_n' &= F_n(t, x_1, x_2, \ldots x_n) \end{aligned}$$

Ex) $n = 1$ $\quad x'(t) = x^2, \quad x(0) = 1$

$$\frac{x'}{x^2} = 1$$

$$\frac{-1}{x} = t + C. \quad C = -1 \text{ since } x(0) = 1$$



$x(t) = \frac{1}{1-t}$ Solution

A solution doesn't always exist!

Ex)

$x'(t) = 3x^{2/3}$ $\qquad x(0) = 0$

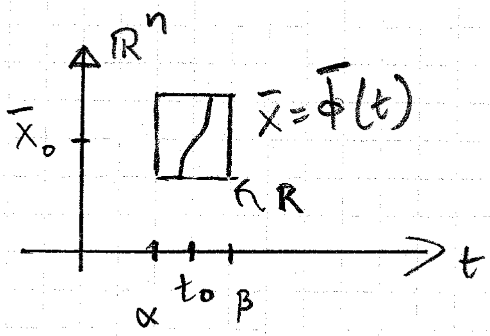
$\frac{x'}{3x^{2/3}} = 1$

$x^{1/3} = t + C = t$

$x = t^3$ is a solution but $x = 0$ is also a solution.

A solution is not always unique!

When are we guaranteed to have a solution which is unique?



In an interval $|t - t_0| < h$ when $F_1, F_2 \ldots F_n$ and $\frac{\partial F_1}{\partial x_1}, \frac{\partial F_1}{\partial x_2} \ldots \frac{\partial F_n}{\partial x_n}$ are continous in a region R

(see theorem 7.1.1)

We come back to existence and uniqueness problem at the end of the course when we consider Picard's method in Sect. 2.8.

For linear systems

$$\bar{X} = P(t)\,\bar{X} + \bar{g}(t)$$

there exists a unique solution as long as $P_{11}, P_{12}, \dots P_{nn}, g_1 \dots$ and g_n are continous, (Th 7.1.2)

7.4

$$\bar{x}' = \underset{\text{matrix}}{P(t)}\,\bar{x} + \underset{\text{vector}}{\bar{g}(t)}$$

If P is a diagonal matrix then we have n problems we already know how to solve

$$x_i'(t) = P_{ii}(t)\,x_i + g_i \qquad i = 1, 2, \dots, n$$

First we consider the homogeneous equation, $\bar{g}(t) = 0$.

$$\bar{x}' = P(t)\,\bar{x}, \quad \bar{x}(t_0) = \bar{x}_0 \ (*)$$

Non homogeneous case is considered in Sect. 7.9

Denote the solutions to $(*)$

$$\bar{x}^{(1)}, \bar{x}^{(2)}, \dots \bar{x}^{(n)}$$

Four theorems about the solutions

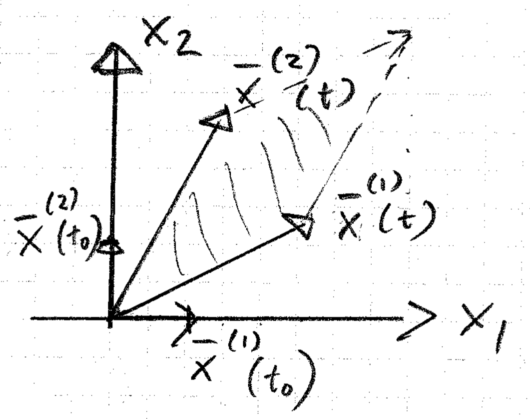
1. $c_1 \bar{X}^{(1)} + c_2 \bar{X}^{(2)}$ also a solution
This is the principle of superposition.

$n=2$ for simplicity

$\bar{X}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ has solution $\bar{X}^{(1)} = \begin{pmatrix} x_{11}(t) \\ x_{21}(t) \end{pmatrix}$

$\bar{X}_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ has solution $\bar{X}^{(2)} = \begin{pmatrix} x_{12}(t) \\ x_{22}(t) \end{pmatrix}$

Phase Space (plane)



Wronskian $W(t) = \begin{vmatrix} x_{11}(t) & x_{12}(t) \\ x_{21}(t) & x_{22}(t) \end{vmatrix}$

2. At $t = t_0$ $\bar{X}^{(1)}$ and $\bar{X}^{(2)}$ are linearly independent so they form a basis. If they remain so in the whole interval $\alpha < t < \beta$ where elements of P are continous then every solution $\bar{\phi}(t)$ can be expressed as a linear combination

$$\bar{\phi}(t) = C_1 \bar{X}^{(1)}(t) + C_2 \bar{X}^{(2)}(t)$$

in exactly one way.

3. In $\alpha < t < \beta$ $W(t)$ is either identical zero or else never vanishes.

4. So in our example above $W(t_0) = 1$ and therefore it remains non-zero in the interval. $\bar{X}^{(1)}$ and $\bar{X}^{(2)}$ form a <u>fundamental set</u> of solutions.

Ex) (7.4.2) Prove that

$$W'(t) = (\mathrm{Tr}\, P)\, W(t)$$

Trace of matrix P

$$\mathrm{Tr}\, P = P_{11}(t) + P_{22}(t)$$

If so

$$W(t) = \underbrace{e^{\int_{t_0}^{t} \mathrm{Tr}\, P}}_{\neq 0} \cdot W(t_0)$$

$$\frac{dW}{dt} = \frac{d}{dt}\left(X_{11} X_{22} - X_{21} X_{12}\right) =$$

$$\begin{vmatrix} \frac{dX_{11}}{dt} & \frac{dX_{12}}{dt} \\ X_{21} & X_{22} \end{vmatrix} + \begin{vmatrix} X_{11} & X_{12} \\ \frac{dX_{21}}{dt} & \frac{dX_{22}}{dt} \end{vmatrix} =$$

$$P_{11} W + P_{22} W = (\mathrm{Tr}\, P)\, W$$

Ex) (7.4.6) $\bar{x}^{(1)} = \begin{pmatrix} t \\ 1 \end{pmatrix}$ $\bar{x}^{(2)} = \begin{pmatrix} t^2 \\ 2t \end{pmatrix}$

a) $W(t) = 2t^2 - t^2 = t^2$

b) The solutions are linearly independent when $t \neq 0$

c) P can not be continous at $t = 0$.

d) $$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} \begin{pmatrix} t \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 2t \\ 2 \end{pmatrix} = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} \begin{pmatrix} t^2 \\ 2t \end{pmatrix}$$

$$\begin{cases} P_{11} t + P_{12} = 1 \\ P_{11} t^2 + 2t P_{12} = 2t \end{cases} \Leftrightarrow \begin{cases} P_{12} = 1 \\ P_{11} = 0 \end{cases}$$

$$\begin{cases} P_{21} t + P_{22} = 0 \\ P_{21} t^2 + P_{22} 2t = 2 \end{cases} \Leftrightarrow \begin{cases} P_{21} = \frac{-2}{t^2} \\ P_{22} = \frac{2}{t} \end{cases}$$