7.5 **Homogeneous Linear Systems with Constant coefficients.**

$$\bar{x}' = A\bar{x}$$

$\det A \neq 0$, then $\bar{0}$ is the only critical point (CP)

If $\lim_{t\to\infty} \bar{x}(t) = \bar{0}$ then the CP is asymptotically stable.

Stable if the solution stay close to $\bar{0}$ for all times.

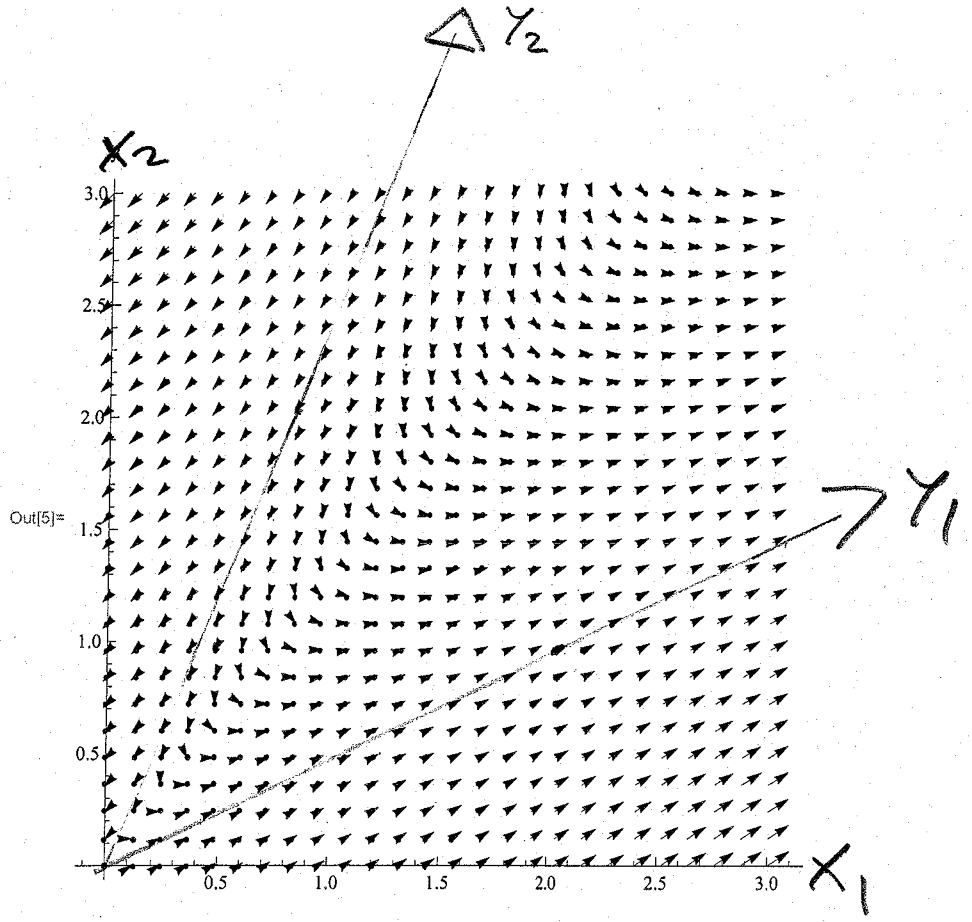
Otherwise the CP is unstable

Ex) 7.5.1
$$\bar{x}' = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3x_1 - 2x_2 \\ 2x_1 - 2x_2 \end{pmatrix}$$

I) Try $\bar{x} = e^{rt}\bar{\varepsilon}$. r must be an eigenvalue to A and $\bar{\varepsilon}$ the corresponding eigenvector. See book

II) Change to new coordinates (functions) $\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$ in such a way that we get two 1D ODE problems. That is

$$\bar{x} = T\bar{y}$$



$$x_1' = 3x_1 - 2x_2$$
$$x_2' = 2x_1 - 2x_2$$

$x_1' = 0$ when $x_2 = \frac{3x_1}{2}$

$x_2' = 0$ when $x_2 = x_1$

$\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ in the Y_1 direction

$\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ in the Y_2 direction

$$\begin{cases} Y_1' = +2Y_1 \\ Y_2' = -Y_2 \end{cases}$$

Inward motion along Y_2-axis

Outward motion along Y_1-axis

$$\bar{x}' = T\bar{y}' = A\bar{x} = AT\bar{y}$$

$$T\bar{y}' = AT\bar{y}$$

$$\bar{y}' = \underbrace{T^{-1}AT}_{D}\,\bar{y}$$

We want D to be a diagonal matrix. Then the columns of T must be the eigenvectors of A.

$$\begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \lambda \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

Non-trivial solution $\iff$ $\begin{vmatrix} 3-\lambda & -2 \\ 2 & -2-\lambda \end{vmatrix} = 0$

$\lambda_1 = 2$ has eigenvector $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$. $\lambda_2 = -1$ has eigenvector $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

$$D = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$$

Easy to solve $\quad y_1' = 2y_1$

$\quad y_2' = -y_2$

The equations don't "talk" to each other.

$$y_1 = c_1 e^{2t}, \quad y_2 = c_2 e^{-t}$$

Back to $\bar{x}$

$$\bar{x} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} c_1 e^{2t} \\ c_2 e^{-t} \end{pmatrix}$$

$$\bar{x} = c_1 e^{2t}\begin{pmatrix}2\\1\end{pmatrix} + c_2 e^{-t}\begin{pmatrix}1\\2\end{pmatrix}$$

Expansion in $\begin{pmatrix}2\\1\end{pmatrix}$ direction

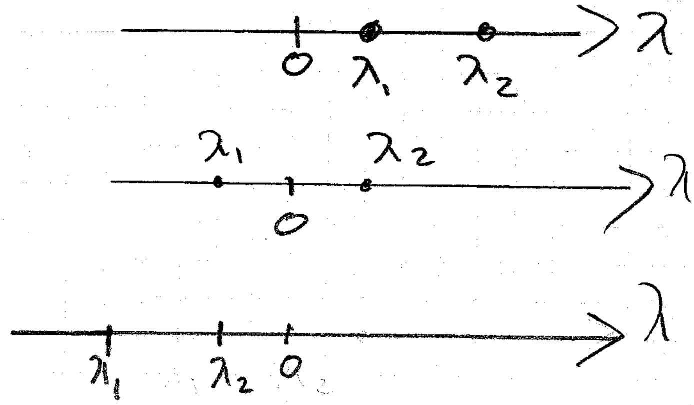
Contraction in $\begin{pmatrix}1\\2\end{pmatrix}$ direction

The CP $\bar{o}$ is here a <u>saddle point</u>.

What happends if the eigenvalues are complex? see 7.6

What happends when we have less than n eigenvectors? see 7.8

So far we have 3 cases:



$\bar{o}$ unstable

$\bar{o}$ saddle point

$\bar{o}$ asymptotically stable.

Ex 7.5.11) $\quad \bar{x}' = \begin{pmatrix}1&1&2\\1&2&1\\2&1&1\end{pmatrix}\bar{x}$

Diagonalisation leads to the eq. $\quad \begin{vmatrix}1-\lambda & 1 & 2\\ 1 & 2-\lambda & 1\\ 2 & 1 & 1-\lambda\end{vmatrix} = 0$

Cubic equation

Note 4 is an eigenvalue with eigenvector $\begin{pmatrix}1\\1\\1\end{pmatrix}$. Then easy to find the other eigenvalues.

$$\bar{x} = c_1 e^{4t}\begin{pmatrix}1\\1\\1\end{pmatrix} + c_2\begin{pmatrix}1\\-2\\1\end{pmatrix}e^{t} + c_3\begin{pmatrix}1\\0\\-1\end{pmatrix}e^{-t}$$

Ex 7.6.1)

$$A = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix}$$

$$\begin{vmatrix} 3-\lambda & -2 \\ 4 & -1-\lambda \end{vmatrix} = 0$$

$$\lambda = 1 \pm i2$$

We denote the eigenvector to $\lambda = 1+2i$ $\bar{a} + i\bar{b}$. One solution to ODE $\bar{x}' = A\bar{x}$ is

$$e^{(1+2i)t}(\bar{a} + i\bar{b}) =$$

$$e^t\left[(\bar{a}\cos 2t - \bar{b}\sin 2t) + i (\bar{a}\sin 2t + b\cos 2t)\right]$$

Real-valued solutions to ODE are

$$\bar{u}(t) = e^t(\bar{a}\cos 2t - \bar{b}\sin 2t)$$

and $\bar{v}(t) = e^t(\bar{a}\sin 2t + \bar{b}\cos 2t)$

$i \to -i$ gives no new solutions.

What is $\bar{a}$ and $\bar{b}$?

$$\begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = (1+2i)\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 1-i \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + i \begin{pmatrix} 0 \\ -1 \end{pmatrix}. \quad \bar{a} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$\bar{b} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$. Finally

$$\bar{u}(t) = e^t \begin{pmatrix} \cos 2t \\ \cos 2t + \sin 2t \end{pmatrix}$$

$$\bar{v}(t) = e^t \begin{pmatrix} \sin 2t \\ \sin 2t - \cos 2t \end{pmatrix}$$

$\bar{0}$ is a <u>spiral point</u> and it is asymptotically unstable.

For example if $\bar{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = c_1 \bar{u}(0) + c_2 \bar{v}(0)$ we get $c_1 = c_2 = 1$

$$\bar{x}(t) = e^t \begin{pmatrix} \cos 2t + \sin 2t \\ 2 \sin 2t \end{pmatrix}$$

see next page

Ex 7.6.28 $\quad m u''(t) + k u(t) = 0$

$x_1 = u$

$x_2 = u'$

Spring without friction

$x_1' = x_2$

$x_2' = u'' = \frac{-k}{m} u = -\frac{k}{m} x_1$

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{bmatrix}$$

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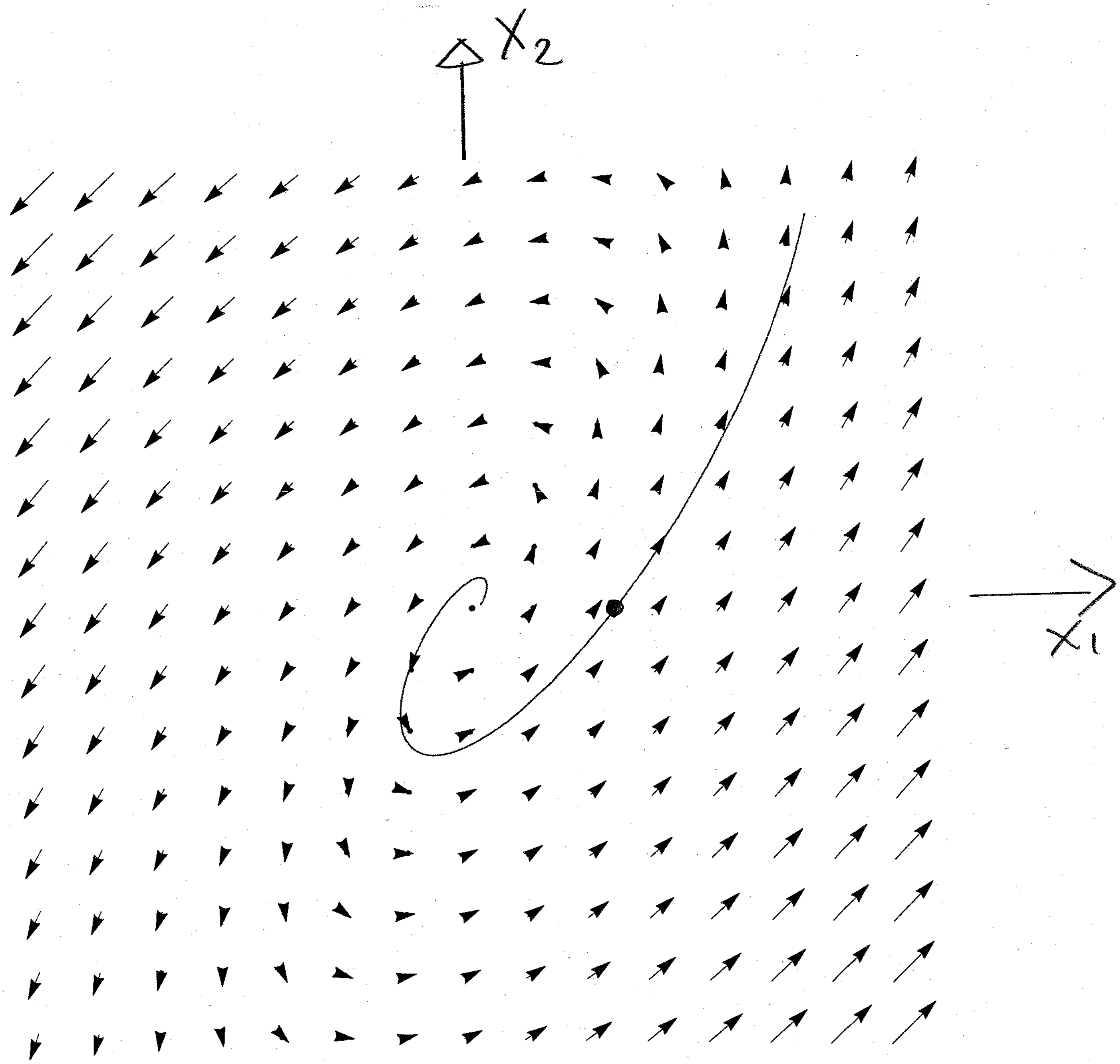


$$\bar{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$-3 \leq t \leq \frac{1}{2}$$

$$\begin{cases} x_1' = 3x_1 - 2x_2 \\ x_2' = 4x_1 - x_2 \end{cases}$$

6,5

Put $k/m = \omega^2$. Solving

$$\begin{vmatrix} -\lambda & 1 \\ -\omega^2 & -\lambda \end{vmatrix} = 0 \iff$$

$$\lambda^2 + \omega^2 = 0 \iff \lambda = \pm i\omega$$

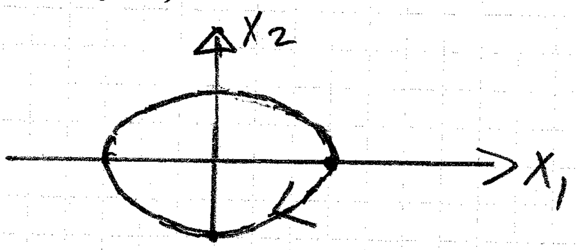
$$\bar{a} + i\bar{b} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + i \begin{pmatrix} 0 \\ \omega \end{pmatrix} \quad \text{check!}$$

$$\bar{u}(t) = \cos\omega t \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \sin\omega t \begin{pmatrix} 0 \\ \omega \end{pmatrix} = \begin{pmatrix} \cos\omega t \\ -\omega\sin\omega t \end{pmatrix}$$

$$\bar{v}(t) = \sin\omega t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \cos\omega t \begin{pmatrix} 0 \\ \omega \end{pmatrix} = \begin{pmatrix} \sin\omega t \\ \omega\cos\omega t \end{pmatrix}$$

$|\bar{u} \times \bar{v}| = \omega$ so $\bar{u}$ and $\bar{v}$ are linearly independent for all times.

If $\bar{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ then $\bar{x}(t) = \bar{u}(t)$



$\bar{0}$ is stable.

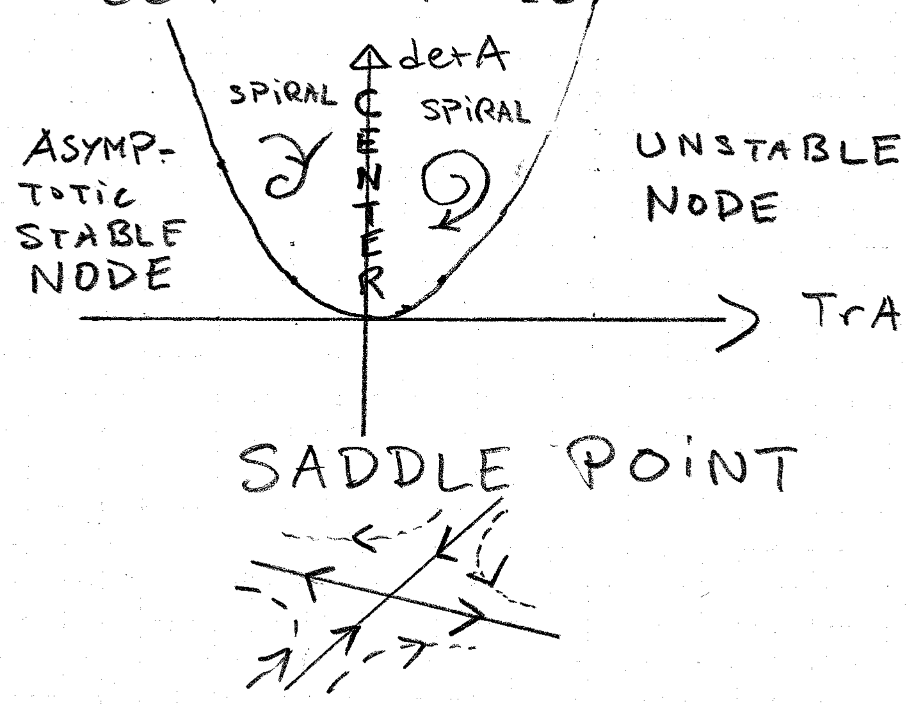
$n=2$ overview

$$\begin{vmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{vmatrix} = 0$$

$$\lambda_{1,2} = \frac{\operatorname{Tr}A \pm \sqrt{(\operatorname{Tr}A)^2 - 4\det A}}{2}$$

Here $\operatorname{Tr} A = a_{11} + a_{22}$ and

$\det A = a_{11} \cdot a_{22} - a_{21} \cdot a_{12}$



On next page you see some illustrations of what can happen in $n=3$ case.

From V. I. Arnold
"ODE"

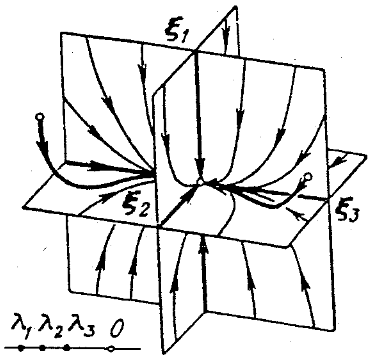


Fig. 137 Phase space of a linear equation in the case $\lambda_1 < \lambda_2 < \lambda_3 < 0$. The phase flow is a contraction in all three directions.

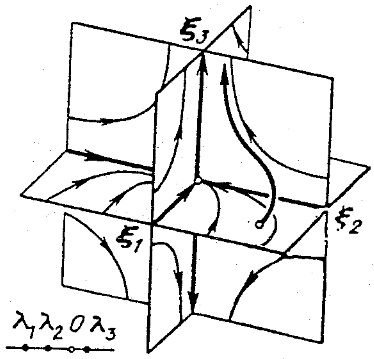


Fig. 138 The case $\lambda_1 < \lambda_2 < 0 < \lambda_3$: Contraction in two directions and expansion in the third.

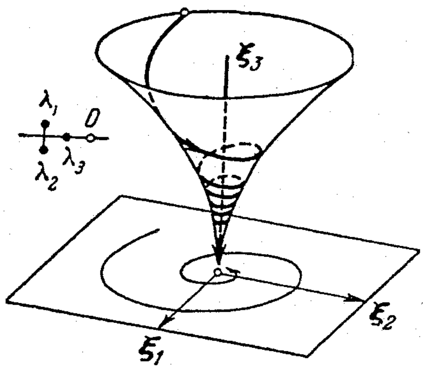


Fig. 139 The case $\mathrm{Re}\,\lambda_{1,2} < \lambda_3 < 0$: Contraction in the direction of ξ_3 and rotation with faster contraction in the plane of ξ_1 and ξ_2.

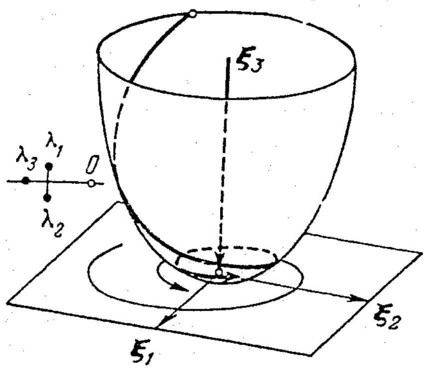


Fig. 140 The case $\lambda_3 < \mathrm{Re}\,\lambda_{1,2} < 0$: Contraction in the direction of ξ_3 and rotation with slower contraction in the plane of ξ_1 and ξ_2.

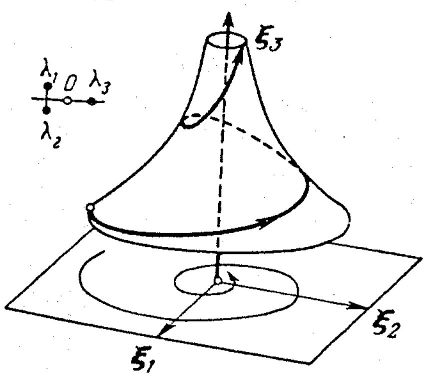


Fig. 141 The case $\mathrm{Re}\,\lambda_{1,2} < 0 < \lambda_3$: Expansion in the direction of ξ_3 and rotation with contraction in the plane of ξ_1 and ξ_2.

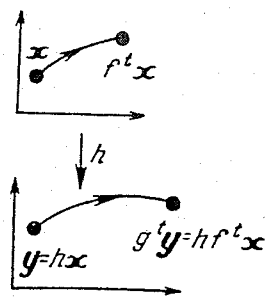


Fig. 142 Equivalent flows.