7.7 – 7.9. Fundamental Matrices, Repeated eigenvalues, Nonhomogeneous Linear Systems.

First $\bar{x}' = P(t)\,\bar{x}$

$$\psi(t) = \begin{bmatrix} x_1^{(1)} & \cdots & x_1^{(n)} \\ \vdots & \vdots & \vdots \\ x_n^{(1)} & \vdots & x_n^{(n)} \end{bmatrix}$$

Fundamental matrix

solutions

$$\det \psi \neq 0$$

General solution $\bar{x} = \psi\,\bar{c} = \psi \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$

With initial value $\bar{x}(t_0) = \bar{x}_0 = \psi(t_0)\,\bar{c}$

we get $\bar{x} = \psi(t)\,\psi^{-1}(t_0)\,\bar{x}_0$

Each column of ψ is a solution to $\bar{x}' = P\bar{x}$ so

$$\psi' = P\psi$$

$\phi(t)$ denotes the special fundamental matrix for which $\phi(0) = I = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$

Then $\boxed{\bar{x} = \phi(t)\,\bar{x}_0}$

e^{At})

$$\bar{x}' = A\bar{x}$$

$$\boxed{\begin{aligned} \phi' &= A\phi \\ \phi(0) &= I \end{aligned}} \quad (*)$$

compare with $x'(t) = ax$, $x(0) = 1$ which has solution $x(t) = e^{at}$

Therefore we introduce e^{At} or $\exp(At)$ for the solution to (*) above.

$$e^{at} = 1 + \sum_{n=1}^{\infty} \frac{a^n t^n}{n!}$$

$$e^{At} \equiv I + At + \frac{A^2 t^2}{2!} + \frac{A^3 t^3}{3!} + \cdots$$

Convergent and OK to differentiate term by term.

$$\frac{d}{dt}\left(e^{At}\right) = A + A^2 t + \frac{A^3 t^2}{2!} + \cdots$$

$$= A\left(I + At + \frac{A^2 t^2}{2!} + \cdots\right) =$$

$$= A e^{At}$$

Ex) $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Determine e^{At}

i) Direct calculation.

$$A^2 = A \cdot A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = I$$

$$A^3 = A^2 \cdot A = I \cdot A = A, \quad A^4 = I \cdot I = I$$

$$e^{tA} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \dots$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix} + \begin{pmatrix} t^2/2 & 0 \\ 0 & t^2/2 \end{pmatrix} + \begin{pmatrix} 0 & t^3/3! \\ t^3/3! & 0 \end{pmatrix} + \dots$$

$$= \begin{pmatrix} \frac{e^t+e^{-t}}{2} & \frac{e^t-e^{-t}}{2} \\ \frac{e^t-e^{-t}}{2} & \frac{e^t+e^{-t}}{2} \end{pmatrix}$$

ii) $A = T^{-1}DT$ D is diagonal.

Note $A^k = (T^{-1}DT)^k = T^{-1}D^kT$

so

$$e^{At} = T^{-1}e^{tD}T =$$

$T^{-1}\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} T$ since A

has eigenvalues ± 1 so $D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$T = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ eigenvector to -1

eigenvector to $+1$

$$T^{-1} = \frac{1}{-2}\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$e^{tA} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} =$$

$$= \frac{1}{2}\begin{pmatrix} e^t + e^{-t} & e^t - e^{-t} \\ e^t - e^{-t} & e^t + e^{-t} \end{pmatrix}$$

iii) Solve the problem in normal way

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^t + c_2\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{-t}$$

Intial values $\bar{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ give first and second column, respectively, of e^{At}. Check!

7.8 What happends when the multiplicity of an eigenvalue λ is ≥ 2 ? No difference compared to previous results if matrix A is symmetric. But in other cases there can be less than n eigenvectors.

Ex 7.8.1 $\bar{x}' = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}\bar{x} = A\bar{x}$

$\operatorname{Tr} A = 2$
$\det A = 1$
$(\operatorname{Tr} A)^2 - 4\det A = 0$ so we are on the border between node and spiral.

$$\lambda = 1 \pm \frac{1}{2}\sqrt{4-4} = 1.$$

eigenvector is $\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \bar{\varepsilon}$.

$\bar{x} = \bar{0}$ is an <u>improper node</u>

$\bar{x}^{(1)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^t$ is one solution

$\bar{x}^{(2)} = \bar{\varepsilon} t e^t + \bar{\eta} e^t$ is the correct ansatz for second solution. .....

But I prefer .... change of coordinates

$$T = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$$

my choice

$\bar{x} = T\bar{y}$

$$\bar{x}' = T\bar{y}' = A\bar{x} = AT\bar{y}$$

$$\bar{y}' = T^{-1}AT\bar{y}$$

This time $T^{-1}AT = \begin{pmatrix} 1 & -5 \\ 0 & 1 \end{pmatrix}$

is not diagonal!

But it is triangular!

$$\begin{cases} y_1' = y_1 - 5y_2 & (1) \\ y_2' = y_2 & (2) \end{cases}$$

$y_2 = C_2 e^t$. Put this into (1)

$$y_1' = y_1 - 5C_2 e^t$$

$$e^{-t}(y_1' - y_1) = -5C_2 e^{-t} \cdot e^t = -5C_2$$

$$D(e^{-t} y_1) = -5C_2$$

$$e^{-t} y_1 = -5C_2 t + C_1$$

$$y_1(t) = C_1 e^t - 5C_2 t e^t$$

Back to (x_1, x_2):

$$\bar{x} = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} \bar{y} =$$

$$c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^t + c_2 \left[\begin{pmatrix} -1 \\ 2 \end{pmatrix} e^t - 5 \begin{pmatrix} 2 \\ 1 \end{pmatrix} t e^t \right]$$

7.9 Non-homgeneous Linear Systems

$$\bar{x}'(t) = P(t)\,\bar{x} + \bar{g}(t)$$

The solution is on the form

$$\bar{X} = \underbrace{\sum_{i=1}^{n} c_i \bar{X}^{(i)}(t)}_{\text{homogeneous solution}} + \underbrace{\bar{V}(t)}_{\text{particular solution}}$$

Various methods exist. Variation of parameters, undetermined coefficients but as usual I prefer diagonalization.

Ex 7.9.1)

$$\bar{x}' = \underbrace{\begin{pmatrix} 3 & -1 \\ 3 & -2 \end{pmatrix}}_{A} \bar{x} + \begin{pmatrix} e^t \\ t \end{pmatrix}$$

$$|A - \lambda I| = 0$$

$$\Leftrightarrow \lambda = \pm 1 \quad . \quad T = \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}.$$

$$T^{-1} = \frac{1}{2}\begin{pmatrix} 3 & -1 \\ -1 & 1 \end{pmatrix}$$

$$\bar{x}' = T\bar{y}' = AT\bar{y} + \bar{g}$$

$$\bar{y}' = \underbrace{T^{-1}AT}_{=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}\bar{y} + T^{-1}\bar{g}$$

Note!

$$\begin{cases} y_1' = y_1 + \frac{1}{2}(3e^t - t) \\ y_2' = -y_2 + \frac{1}{2}(-e^t + t) \end{cases}$$

Multiply with IF, do partial integrations.

$$Y_1(t) = C_1 e^{t} + \frac{3t}{2} e^{t} + \frac{(t+1)}{2}$$

$$Y_2(t) = C_2 e^{-t} - \frac{1}{4} e^{t} + \frac{(t-1)}{2}$$

Finally $\quad \bar{x} = T\bar{y}, \quad T = \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}$

$$\bar{x} = \underbrace{c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t} + c_2 e^{-t} \begin{pmatrix} 1 \\ 3 \end{pmatrix}}_{\text{Homogeneous solution}} + \underbrace{\frac{e^{t}}{2} \begin{pmatrix} 3t - 1/2 \\ 3t - 3/2 \end{pmatrix} + \begin{pmatrix} t \\ 2t-1 \end{pmatrix}}_{\text{Particular solution}}$$

For an initial value problem it is now time to determine constants c_1 and c_2.