Chapter 9 Nonlinear Differential Equations and Stability

TABLE 9.1.1 Stability Properties of Linear Systems $\mathbf{x}' = \mathbf{A}\mathbf{x}$ with $\det(\mathbf{A} - r\mathbf{I}) = 0$ and $\det \mathbf{A} \neq 0$

Eigenvalues	Type of Critical Point	Stability
$r_1 > r_2 > 0$	Node	Unstable
$r_1 < r_2 < 0$	Node	Asymptotically stable
$r_2 < 0 < r_1$	Saddle point	Unstable
$r_1 = r_2 > 0$	Proper or improper node	Unstable
$r_1 = r_2 < 0$	Proper or improper node	Asymptotically stable
$r_1, r_2 = \lambda \pm i\mu$	Spiral point	
$\lambda > 0$		Unstable
$\lambda < 0$		Asymptotically stable
$r_1 = i\mu, r_2 = -i\mu$	Center	Stable

Ex 9.1.17

$$mu''(t) + cu'(t) + ku(t) = 0$$

$m > 0$

$c > 0$

$k > 0$

Spring with damping.

$F = -ku - cu'$

$ma = mu'' = F$

$x = u(t)$

$y = u'(t)$

$$\frac{dx}{dt} = u' = y$$

$$y' = u'' = -\frac{c}{m}u' - \frac{k}{m}u =$$

$$-\frac{k}{m}x - \frac{c}{m}y$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$\mathrm{Tr}\,A = -c/m$ $\qquad$ $\det A = k/m$

$$\lambda_{1,2} = \frac{\mathrm{Tr}A \pm \sqrt{(\mathrm{Tr}A)^2 - 4\det A}}{2}$$

$\operatorname{Tr} A < 0$ and $\det A > 0$ so $\bar{0}$ is a asymptotically stable CP.

If $(\operatorname{Tr} A)^2 < 4 \det A$ $\bar{0}$ is a <u>spiral sink</u>.

$$\frac{c^2}{m^2} < 4 \cdot \frac{k}{m}$$

$$c < 2\sqrt{mk}$$

If $c^2 = 4km$ $\bar{0}$ is an <u>improper node</u>

If $c^2 > 4km$ $\bar{0}$ is an node.

9.2 <u>Autonomous</u> Systems and Stability

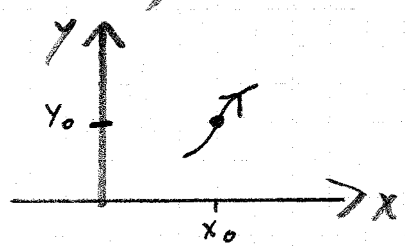
$n = 2$

$$x'(t) = F(x, y)$$
$$y'(t) = G(x, y)$$

or $\frac{d\bar{x}}{dt} = \bar{f}(\bar{x})$

$F, G, \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial G}{\partial x}, \frac{\partial G}{\partial y}$ are continous.

Then unique solution exists (for a while!)



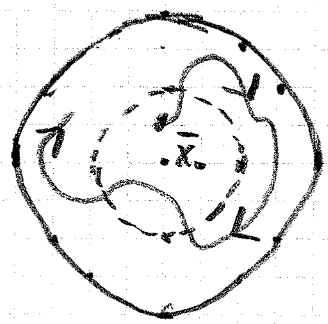
The points where $\bar{f}(\bar{x}) = \bar{0}$ are called <u>critical points</u>.

A CP is

STABLE if for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\|\bar{\phi}(t) - \bar{X}_0\| < \varepsilon \text{ for all}$$

times if $\|\bar{\phi}(0) - \bar{X}_0\| < \delta$



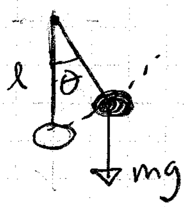
ASYMPTOTICALLY STABLE if there exists a δ such that

$$\|\bar{\phi}(0) - \bar{X}_0\| < \delta \implies$$

$$\lim_{t \to \infty} \bar{\phi}(t) = \bar{X}_0$$

Otherwise the CP is unstable

Ex) 9.2.21
Pendulum without friction



$v = l\dot{\theta}$
$a = l\ddot{\theta}$

Newton's eq:

$$F = ma$$

$$-mg\sin\theta = ml\,\theta''(t)$$

$$\theta'' = -\frac{g}{l}\sin\theta \qquad \text{put } g/l = 1$$

Go over to a system.

$$x = \theta \;,\; y = \theta'$$

$$\begin{cases} x' = y \\ y' = -\sin x \end{cases}$$

$(0,0)$ is a CP - pendulum hanging down.

$(\pm\pi, 0)$ are also critical points — pendulum upside down.

Note $\quad \dfrac{dy}{dx} = -\dfrac{\sin x}{y} \iff y\,y'(x) = -\sin x$

So the motion lies on curves

$$\frac{y^2}{2} - \cos x = C$$

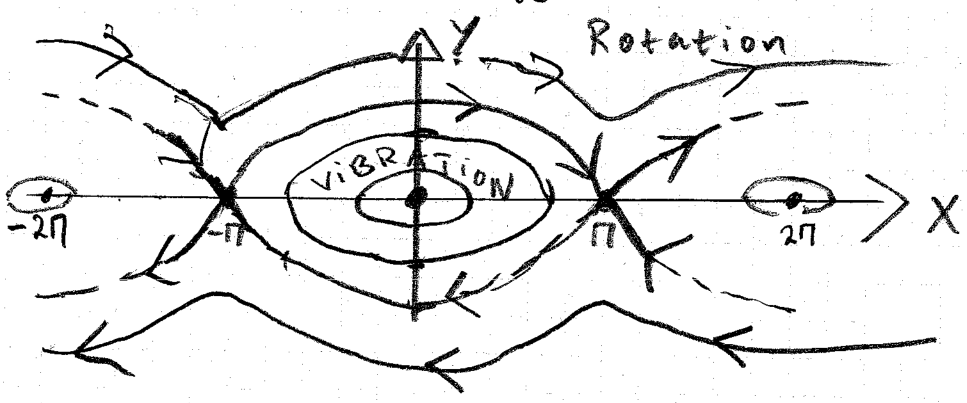
$$y^2 = 2C + 2\cos x = 2(C + \cos x)$$

Since energy is conserved for the pendulum

$$E = \frac{m(l\theta')^2}{2} + mgl(1 - \cos\theta)$$

is a constant

becomes $\quad \dfrac{E}{m} - 1 = \dfrac{y^2}{2} - \cos x \quad$ when $\dfrac{g}{l} = 1$



9.3 Almost Linear Systems (ALS)

$\bar{x} = \bar{0}$ is a CP.

Definition of an ALS:

$$\bar{x}' = A\bar{x} + \bar{g}(\bar{x}), \quad \det A \neq 0$$

$\bar{g}$ is small in the following sense

$$\lim_{\bar{x} \to \bar{0}} \|\bar{g}(\bar{x})\| / \|\bar{x}\| = 0$$

If we find a CP for an ALS, can we trust the linear system? That is, will the CP have the same character in the LS as in the ALS?

Yes, except for centers and improper nodes. Proof in 9.6

TABLE 9.3.1 Stability and Instability Properties of Linear and Almost Linear Systems

	Linear System		Almost Linear System	
r_1, r_2	Type	Stability	Type	Stability
$r_1 > r_2 > 0$	N	Unstable	N	Unstable
$r_1 < r_2 < 0$	N	Asymptotically stable	N	Asymptotically stable
$r_2 < 0 < r_1$	SP	Unstable	SP	Unstable
$r_1 = r_2 > 0$	PN or IN	Unstable	N or SpP	Unstable
$r_1 = r_2 < 0$	PN or IN	Asymptotically stable	N or SpP	Asymptotically stable
$r_1, r_2 = \lambda \pm i\mu$				
$\lambda > 0$	SpP	Unstable	SpP	Unstable
$\lambda < 0$	SpP	Asymptotically stable	SpP	Asymptotically stable
$r_1 = i\mu, r_2 = -i\mu$	C	Stable	C or SpP	Indeterminate

Note: N, node; IN, improper node; PN, proper node; SP, saddle point; SpP, spiral point; C, center.

Ex) 9.3.5 $\frac{dx}{dt} = (2+x)(y-x)$

$\frac{dy}{dt} = (4-x)(y+x)$

First eq is zero when $x=-2$ and $y=x$

Second eq is zero when $x=4$ and $y=-x$

So the CP's are

$(-2,2)$, $(4,4)$ and $(0,0)$.

$(-2,2)$: $x=-2+u$ $\quad x'=u'$
$y=2+v$ $\quad y'=v'$

The ODE's in (u,v) variables:

$u' = 4u - u^2 - uv$
$v' = 6u + 6v - u^2 - uv$

$A = \begin{pmatrix} 4 & 0 \\ 6 & 6 \end{pmatrix}$. $\frac{\| u^2 + uv \|}{r} = \frac{r^2 \| \cos^2\varphi + \cos\varphi \sin\varphi \|}{r}$

$\to 0$ when $r \to 0$ so it is an ALS.

$\mathrm{Tr}\, A = 10$
$\det A = 24$

$\lambda_{1,2} = \frac{10 \pm \sqrt{10^2 - 4 \cdot 24}}{2} = 4$ and 6

Conclusion $(-2,2)$ is an unstable node.

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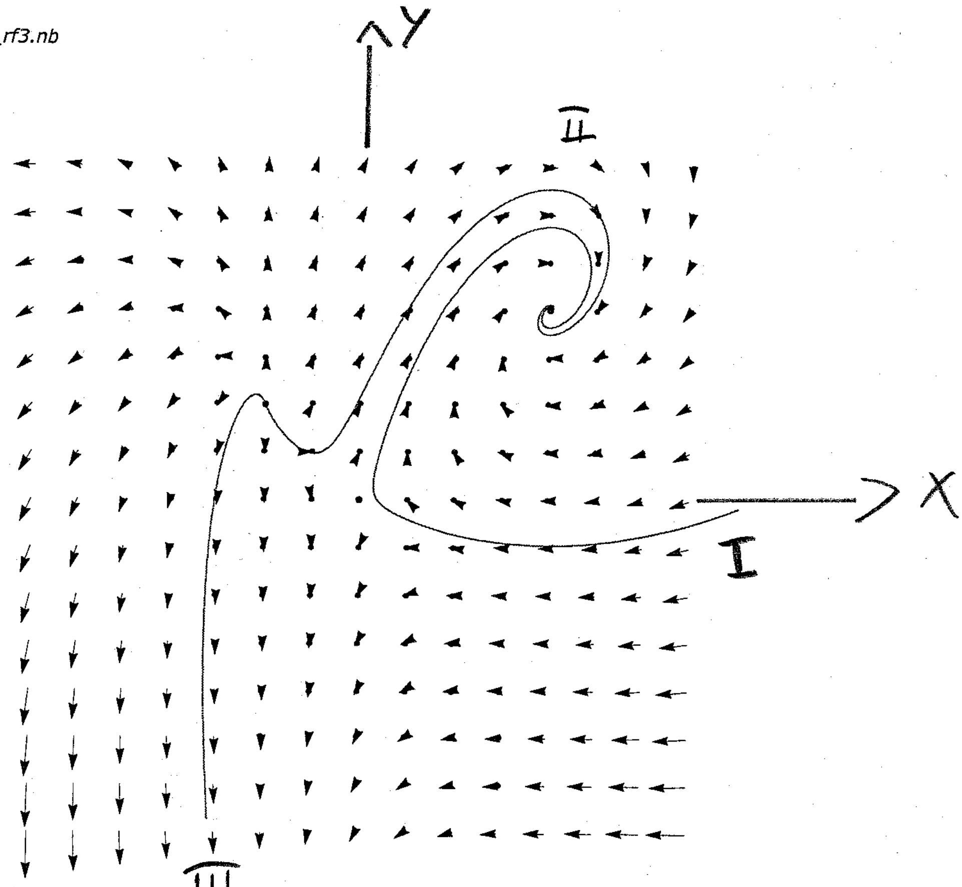

(-2,2), (0,0) and (4,4) are the critical points.

Three orbits are shown

I) (x,y) @ (0.3, 0.3) for t=0.

$-0.8 \le t \le 3$

II) (x,y) @ (4.5, 3.7) for t=0.

$-3 \le t \le 3$

III (x,y) @ (-2.2, 2.2) for t=0.

$-1.1 \le t \le 0.55$