9.6-9.7 **Liapunov's Second Method + Periodic Solutions and Limit Cycles**

For a pendulum with engery E = kinetic + potential energy

$\bar{0}$ As. St. $\quad \frac{dE}{dt} < 0$ with friction

$\bar{0}$ a center $\quad \frac{dE}{dt} = 0$ without friction

---

Idea: You expect that $\bar{0}$ is As. St. Then find function $V(x,y)$ such that

1) $V(x,y) > 0 \quad (x,y) \neq (0,0)$

2) $\dot{V} < 0$

Here $x' = F(x,y)$, $y' = G(x,y)$ and

$$\dot{V} = \frac{dV}{dt} = \frac{\partial V}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial V}{\partial y}\frac{dy}{dt} =$$

$$V_x F(x,y) + V_y G(x,y)$$

1) and 2) hold in a region D containing $\bar{0}$.

If 1) holds V is <u>positive definite</u>

If 2) holds $\dot{V}$ is <u>negative definite</u>.

Ex/ 9.6.1

$$x'(t) = -x^3 + xy^2 = -x(x^2-y^2)$$
$$y'(t) = -2x^2y - y^3 = -y(2x^2+y^2)$$

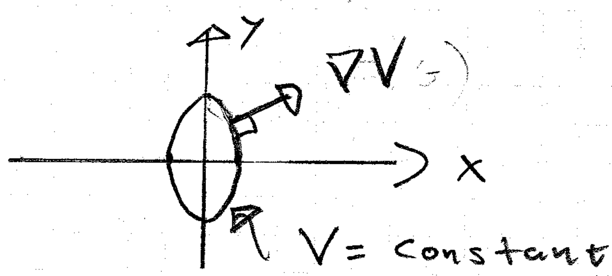
Try $V(x,y) = ax^2 + by^2$ $\quad a>0$, $b>0$

$\bar{0}$ is the CP.

$$\dot{V} = \frac{dV}{dt} = 2ax(-x^3+xy^2) + 2by(-2x^2y-y^3) =$$
$$-2ax^4 + x^2y^2\underbrace{(2a-4b)}_{=0} - 2by^4$$

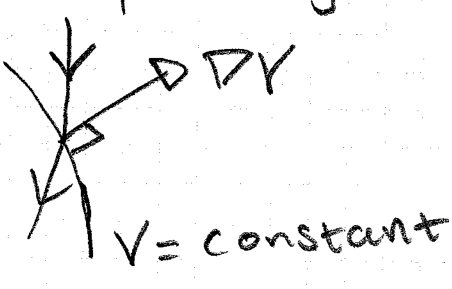
Choose $a=2$ and $b=1$. Then

$$\dot{V} = -(4x^4+2y^4) < 0$$



$$\dot{V} = \nabla V \cdot (x', y') < 0$$

$\bar{x}'$ is always pointing inwards!



V is a <u>strict Liapunov function</u>.

If $\dot{V} \le 0$ in the region outside $\bar{0}$ it is a Liapunov function.

Some Quadratic Forms:

$V(x,y) = x^2 + y^2$ positive definite

$V(x,y) = x^2$ positive semi-definite

$V(x,y) = x^2 - y^2$ Indefinite

$V(x,y) = x^2 + y^2 - \frac{6}{5}xy =$

$$(x,y) \underbrace{\begin{pmatrix} 1 & -3/5 \\ -3/5 & 1 \end{pmatrix}}_{Q} \begin{pmatrix} x \\ y \end{pmatrix} \qquad Q^t = Q$$

Find transformation $\begin{pmatrix} x \\ y \end{pmatrix} = T \begin{pmatrix} u \\ v \end{pmatrix}$ such that $T^{-1} Q T$ becomes diagonal.

The eigenvalues of Q are $2/5$ and $8/5$ so V is positive definte.

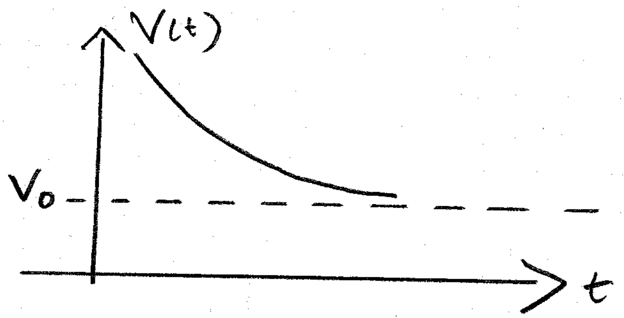
Th 9.6.4: $V(x,y) = ax^2 + bxy + cy^2$

V positive definite $\iff$ $a > 0$, $4ac - b^2 > 0$

V negative definite $\iff$ $a < 0$, $4ac - b^2 > 0$

Some remarks.

1)



Can it happen?

No then there exists a limit cycle (see 9.7) for which $V = V_0$. Impossible since $\dot{V} < 0$ in the region we consider

2) What to do if you suspect that $\bar{o}$ is an unstable CP?

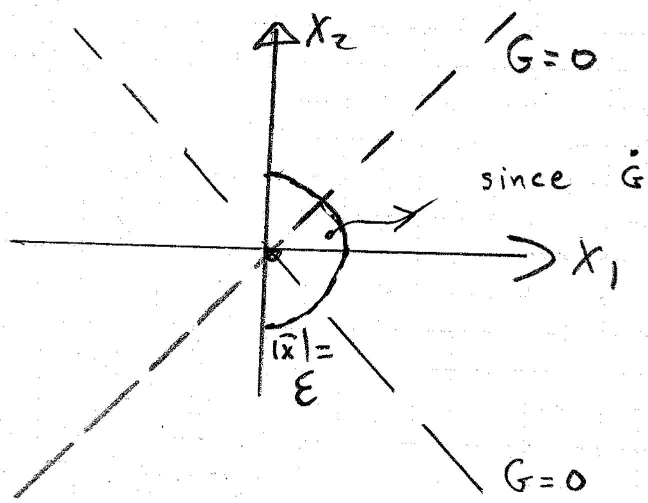
$$\begin{cases} x_1' = x_1 + x_1 x_2 \\ x_2' = -2x_2 + x_1 \cdot x_2 \end{cases}$$

$\bar{o}$ a saddle point?

Introduce $G(x_1, x_2) = x_1^2 - x_2^2$

then $\dot{G} = 2x_1^2 + 4x_2^2 + 2x_1x_2(x_1 - x_2)$

$\dot{G} > 0$ near $\bar{o}$

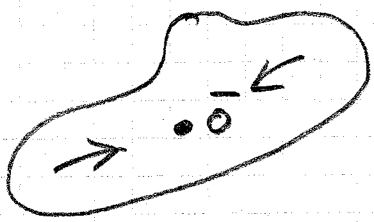


since $\dot{G} > 0$ the orbit must leave the $|\bar{x}| = \varepsilon$ circle

3) If $\bar{o}$ is asymptotically stable the starting points for which

$$\lim_{t \to \infty} \bar{x}(t) = \bar{o}$$

is called the <u>basin of attraction</u> for $\bar{o}$



4) How to construct Liapunov function for ALS - system?

Ex)

$$\frac{dx}{dt} = -2x - 3y + F_1(x,y)$$

$$\frac{dy}{dt} = x + y + G_1(x,y)$$

$\mathrm{Tr} A = -1$, $\det A = 1$.

For the LS $\bar{o}$ is an asymptotically stable spiral node.

With $V(x,y) = \frac{3}{2}x^2 + 5xy + 7y^2$

$\dot{V} = -(x^2+y^2)$ for LS. Check

See Ex 9.6.10 and Ex 9.6.11.

For the ALS

$$\dot{V} = -(x^2+y^2) + (3x+5y)F_1 + (5x+14y)G_1$$

Using polar coordinates $x = r\cos\theta$, $y = r\sin\theta$

and remember that

$$\left|\frac{F_1}{r}\right| \text{ and } \left|\frac{G_1}{r}\right| \to 0 \quad \text{when } r \to 0$$

we see that V is still a strict Liapunov function even for the ALS-system

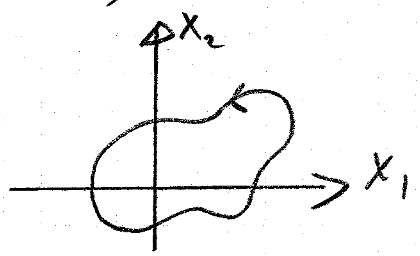
---

9.7. Periodic Solutions and Limit Cycles.

$$\bar{x} = \bar{f}(\bar{x})$$

$$\bar{x}(t+T) = \bar{x}(t) \quad \text{for}$$

a periodic orbit.

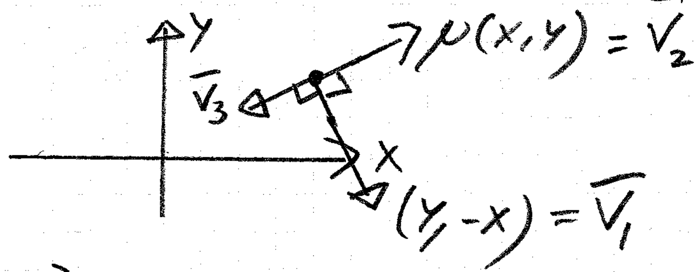


T is the period.

Ex 9.7.16)

$$x'(t) = y + \mu x - x \cdot r^2 \qquad \mu \in \mathbb{R}$$

$$y'(t) = -x + \mu y - y \cdot r^2$$



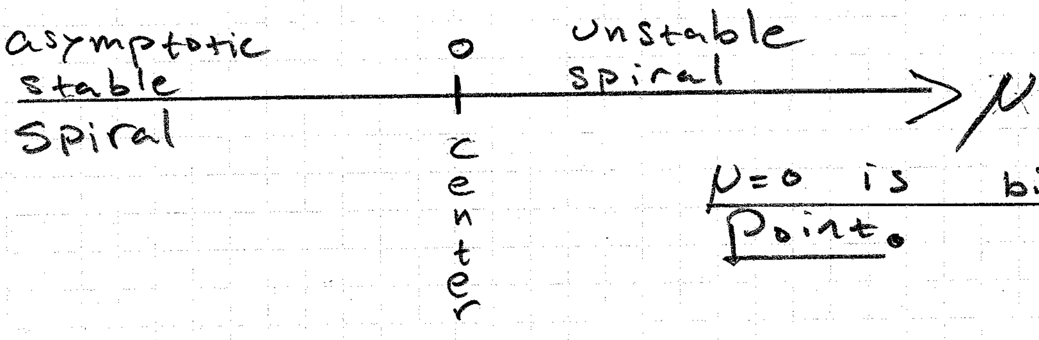
$$\bar{V}_3 = r^2(-x, -y)$$

$\bar{V}_2$ can cancel $\bar{V}_3$ but

$\bar{V}_1$ is $\perp$ to $\bar{V}_2$ and $\bar{V}_3$.

So $\bar{0}$ is the only CP.

$A = \begin{pmatrix} \mu & 1 \\ -1 & \mu \end{pmatrix}$. $\lambda = \mu \pm i$



$\mu = 0$ is bifurcation point.

When $\mu > 0$ there is a limit cycle at $r = \sqrt{\mu}$. There

$\bar{V}_2 + \bar{V}_3 = \bar{0}$.

Becomes clear when using polar coordinates

$$X = r\cos\theta$$
$$y = r\sin\theta$$

The system of ODE:s becomes

$$r' = r(\mu - r^2)$$
$$\theta' = -1$$