Applications

I. Competing Species:

$x(t), y(t)$ two similar species

$$\frac{dx}{dt} = x(\varepsilon_1 - \sigma_1 x) - \alpha_1 x \cdot y$$

$$\frac{dy}{dt} = y(\varepsilon_2 - \sigma_2 y) - \alpha_2 y \cdot x$$

competition!

$\varepsilon_1, \varepsilon_2, \sigma_1, \sigma_2, \alpha_1, \alpha_2$ are positive constants.

Often in nature competition leads to complete extinction of one of the species

"Principle of competitive exclusion" (M. Braun D.Eq. and their Applications)

Let us rewrite the system

$$x'(t) = r_1 x \left(1 - \frac{x}{K_1} - \alpha_{12} \frac{y}{K_1}\right)$$

$$y'(t) = r_2 y \left(1 - \frac{y}{K_2} - \alpha_{21} \frac{x}{K_2}\right)$$

still positive constants

Separate <u>niches</u> $\alpha_{12} = \alpha_{21} = 0$

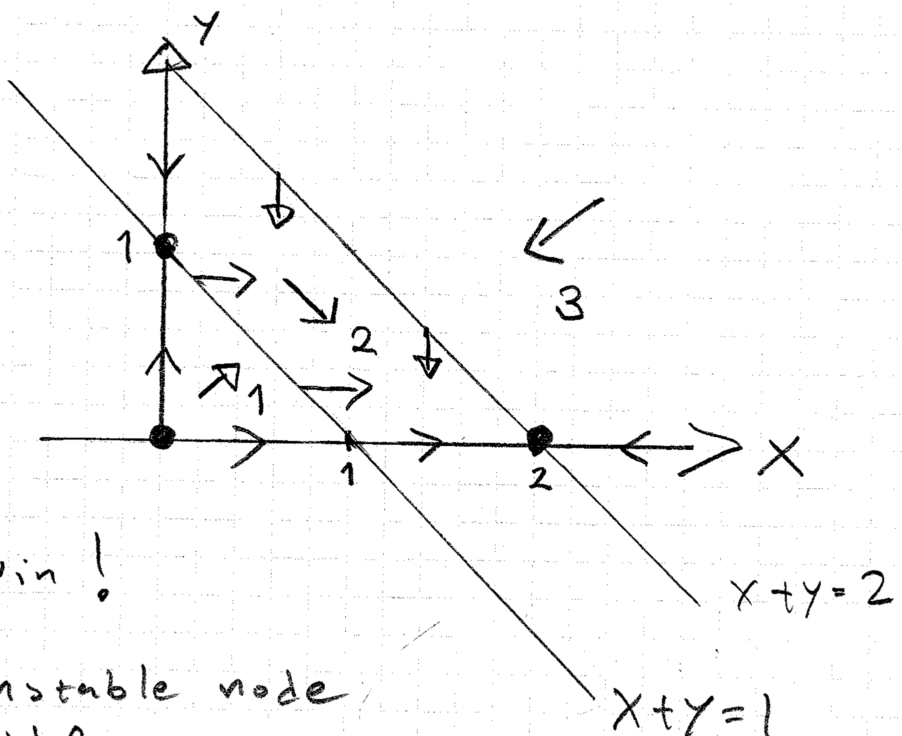
Intense struggle $\alpha_{12} = \alpha_{21} = 1$

Without competition $x(t) \to K_1$ and $y(t) \to K_2$ when $t \to \infty$.

Ex)
$$x'(t) = x\left(1 - \frac{x}{2} - \frac{y}{2}\right)$$
$$y'(t) = y\,(1 - y - x)$$
$$x \geq 0,\ y \geq 0$$

$r_1 = r_2 = 1$
$K_1 = 2$
$K_2 = 1$
$\alpha_{12} = \alpha_{21} = 1$

Critical points $(0,0)$, $(0,1)$ and $(2,0)$



x will win!

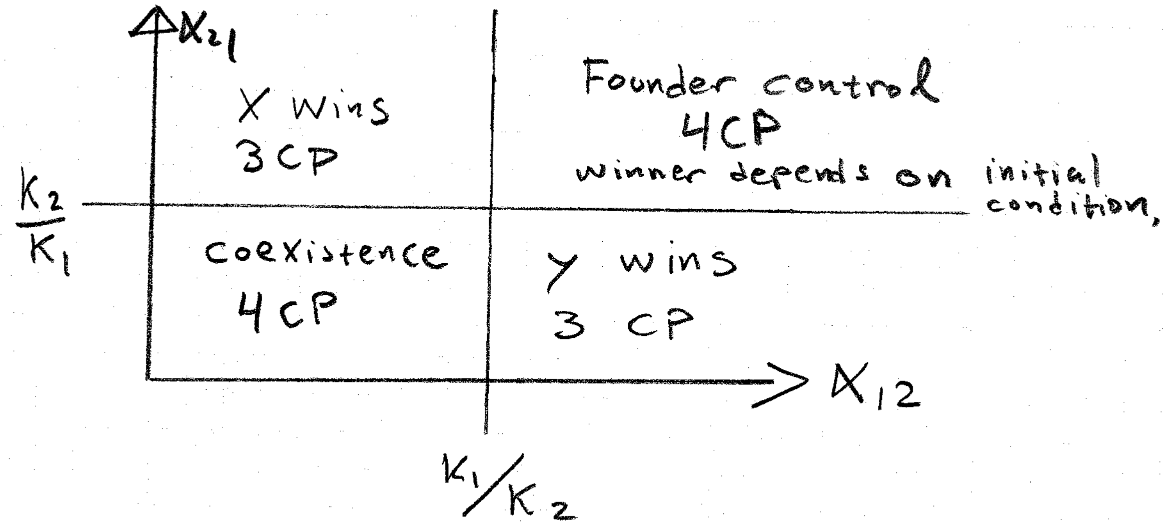
$(0,0)$ an unstable node
$(0,1)$ a saddle point
$(2,0)$ an asymptotic stable node

$x = 2 + u$
$y = v$

L.S. $u'(t) = -u - v$
$v'(t) = -v$

$\operatorname{Tr} A = -2$, $\det A = 1$.

The following figure shows the different possibilities



II Predator – Prey

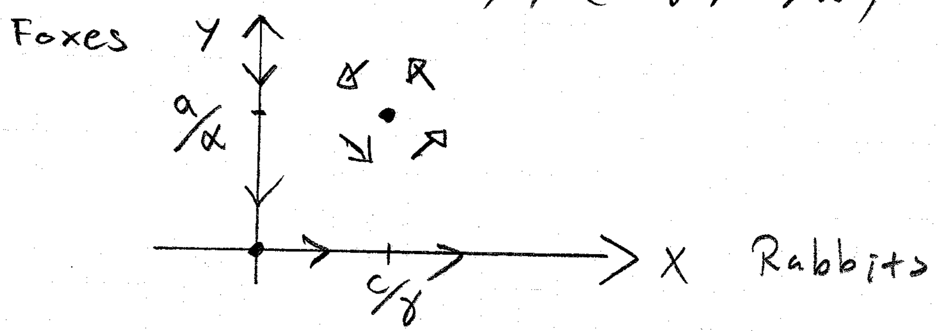
Lotka, Volterra 1925

$$\frac{dx}{dt} = ax - \alpha xy$$

$$\frac{dy}{dt} = -cy + \gamma xy$$

a, c, α, γ are positive constants.

Critical points: $(0,0), (c/\gamma, a/\alpha)$



Ex) $a = c = \alpha = \gamma = 1$

$$x' = x(1-y)$$

$$y' = -y(1-x)$$

Nature of $(1,1)$?

$x = 1 + u$

$y = 1 + v$ $\Rightarrow$ $u'(t) = -v$, $v'(t) = u$.

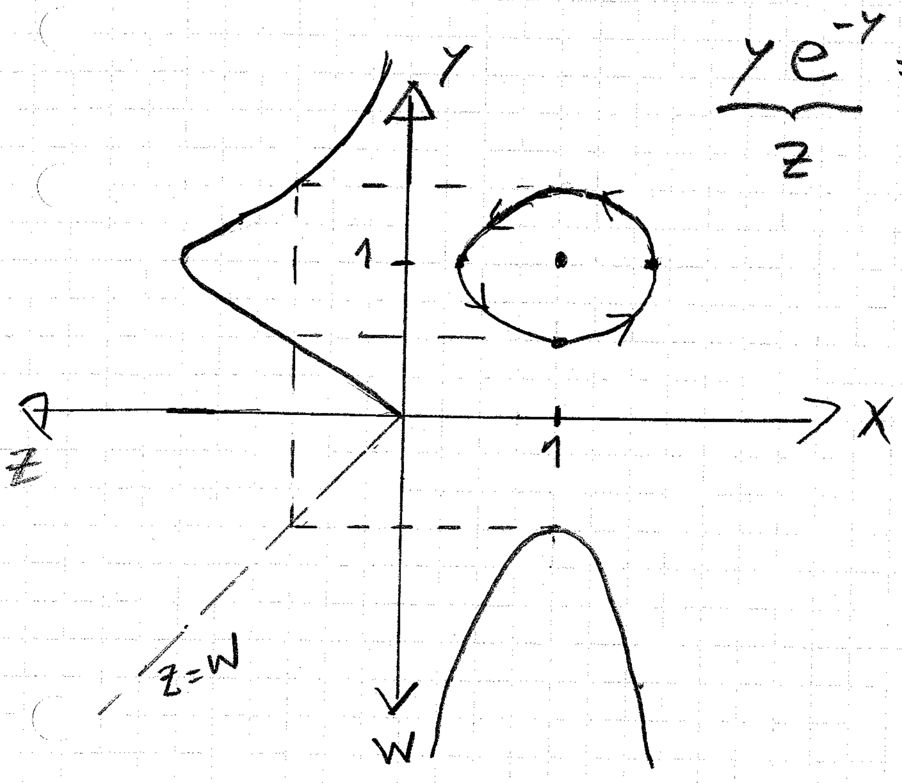
$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} . \quad \lambda^2 = -1 , \quad \lambda = \pm i$$

A center! A more detailed analysis is needed.

Continuation: $y'(x) \cdot \frac{1-y}{y} = \frac{x-1}{x}$

$$\ln y - y = x - \ln x + C_1$$

$$\underbrace{y e^{-y}}_{z} = \underbrace{c_2 \frac{e^x}{x}}_{w}$$



(1,1) also stable for the ALS.

Ex)

year	sharks in % of total catch
1914	11,9
1915	21.4
1916	22.1
1917	21,2
1918	36,4
1919	27.3
1920	16.0
1921	15.9
1922	14.8

Why is a reduced level of fishing <u>more</u> beneficial to the predators than to the prey?

Equations
with fishing :

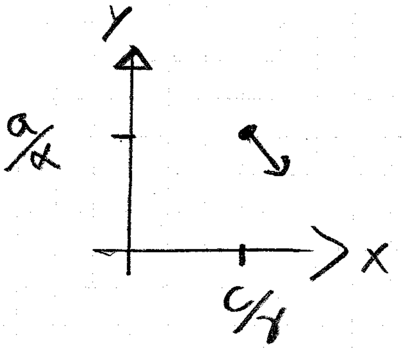
$$x' = x\left[a - \alpha y - \varepsilon\right]$$
$$y' = y\left[-c + \gamma x - \varepsilon\right]$$

moderate
fishing, $\varepsilon \ll a$,

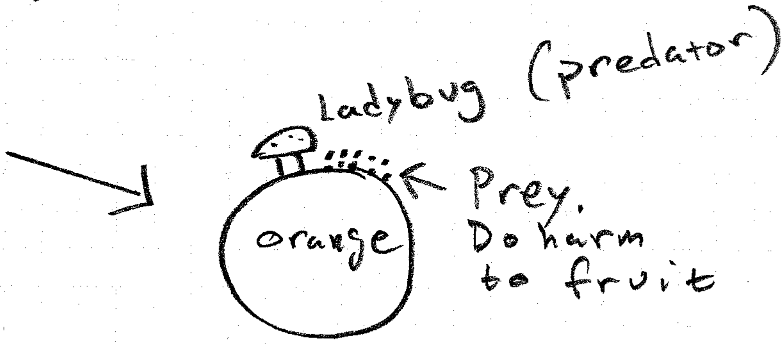
For CP

$$Y_{cp} = (a-\varepsilon)/\alpha$$

$$X_{cp} = \quad X = \frac{c+\varepsilon}{\gamma}$$



Ex) DDT

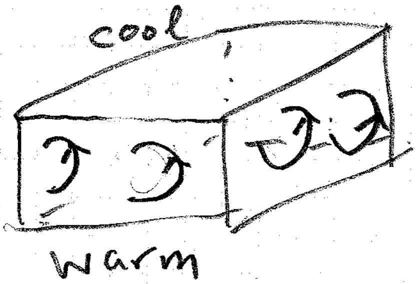


DDT → more
harm to predator!

III Strange attractors

Lorentz
1963

$$x'(t) = \sigma(-x+y) \quad (1)$$
$$y'(t) = rx - y - xz \quad (2)$$
$$z'(t) = -bz + xy \quad (3)$$



ΔT horisontal $= y$
ΔT vertical $= z$
x – intensity of
flow

$\sigma = 10$
$b = 8/3$ for earth's atmosphere

$r \geq 0$, $r \sim \Delta T$.

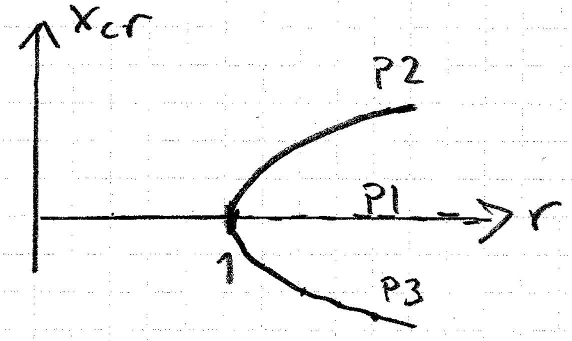
Critical points.

(1) $x = y$

(2') $x[r-1-z] = 0$ $\nearrow$ $x = 0 = y = z$ P1

$\searrow$ P2, P3 $\begin{cases} z = r-1 \\ x = \pm\sqrt{b(r-1)} \\ y = \pm\sqrt{b(r-1)} \end{cases}$ when $r > 1$

(3') $bz = x^2$



Bifurcation at $r = 1$

Near $\bar{0}$: $A = \left(\begin{array}{cc|c} -10 & 10 & 0 \\ r & -1 & 0 \\ \hline 0 & 0 & -\frac{8}{3} \end{array}\right)$ $\lambda_3 = -8/3$

$\lambda_{1,2} = \dfrac{-11 \pm \sqrt{81 + 40r}}{2}$. All eigenvalues negative when $r < 1$. $\bar{0}$ unstable for $r > 1$.

P2 and P3 become unstable for $r > 24.737$. However, all solutions remain bounded when $t \to \infty$. The limit cycle is a <u>strange</u> attractor.

Lorentz attractor