Lecture 13

Singular SL-problems and Bessel's equation   11.4 – 11.5

$$-[p(x)y']' + q(x)y = \lambda r(x) y$$

$$0 < x < 1$$

$p(x) > 0$, $r(x) > 0$ in the closed interval. q, r continous, p differentiable.

Singular SL-problems:

1) infinite interval
2) Relax at least one of the constraints on p, r, q at an endpoint.

What happends then?

i) The spectra can be continous, discrete + continous, discrete and finite number of eigenvalues.

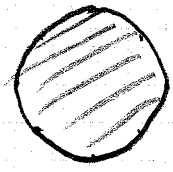
ii) y, y' bounded as $x \to 0$

iii) Improper integrals can appear. Is

$$\lim_{\varepsilon \to 0} \int_\varepsilon^1 (LU)V - U(LV)\,dx = 0 \;?$$

Ex)

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 0$$



$x^2 + y^2 < 1, \quad t > 0$

Polar coordinates $\quad x = r\cos\theta$

$\qquad\qquad\qquad\qquad y = r\sin\theta$

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right) = 0$$

$r < 1, \quad t > 0$

Boundary conditions : $\quad u(t, 1, \theta) = 0, \quad t > 0$

IV: $\quad u(0, r, \theta) = f(r)$

$\qquad\qquad\qquad\qquad\qquad r < 1$

$\qquad \dfrac{\partial u}{\partial t}(0, r, \theta) = 0$

Fouriers method : $u(t, r, \theta) = F(r,\theta)\,T(t)$

gives

$$\frac{1}{c^2}\frac{T''}{T} = \frac{1}{F}\left(F''_{rr} + \frac{1}{r}F'_r + \frac{1}{r^2}F''_{\theta\theta}\right) =$$

$$-\lambda, \qquad \lambda > 0 \quad (\text{oscillations})$$

(1) $\qquad T'' = -\lambda c^2 T$

(2) $\qquad F''_{rr} + \frac{1}{r}F'_r + \frac{1}{r^2}F''_{\theta\theta} = -\lambda F$

second separation $F(r,\theta) = R(r)\,V(\theta)$

$\Rightarrow$

$$r^2 \frac{R''}{R} + r\frac{R'}{R} + \lambda r^2 = -\frac{V''}{V} = m^2$$

m must be an integer, $m = 1, 2, 3 \ldots$, since the solutions to

$$V'' = -m^2 V$$

must have period equal to 2π.

Left to solve

$$r^2 R'' + r R' - m^2 R = -\lambda r^2 R$$

BC: $R(1) = 0$ and

Bounded near $r = 0$

change of variable:

$$S = r\sqrt{\lambda}$$

gives ODE

$$s^2 R''(s) + s R'(s) + (s^2 - m^2) R = 0$$

Bessel's equation (see 5.8)

C

5.8

In 11.4 we ended with

$$s^2 R''(s) + s R'(s) + (s^2 - m^2) R(s) = 0$$

$$s = r\sqrt{\lambda} \qquad \lambda \geq 0$$

r is the radius
m - integer

in 5.8 Bessels equation looks like

$$x^2 y''(x) + x y'(x) + (x^2 - v^2) y(x) = 0$$

Now v is a real constant

Now we know how to solve it!

Some cases

Indicial equation:

1) $v = 0$ $\qquad r \cdot (r-1) + r = 0$

$r = 0$ double zero

$$y_1(x) = a_0 \underbrace{\left[1 + \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m} (m!)^2} \right]}_{= J_0(x)}$$

The other solution, with logaritmic singularity, is usually denoted

$Y_0(x)$ see p. 296-297

For large x: $J_0(x) \sim \sqrt{\frac{2}{\pi x}} \cos(x - \pi/4)$

$$Y_0(x) \sim \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{\pi}{4}\right)$$

2) $\nu = 1/2$ For wave equation in 3 dimensions and spherical coordinates this value occurs.

$$r(r-1) + r - 1/4 = 0$$

$$r_1 = 1/2 \qquad r_2 = -1/2$$

The two solutions are denoted

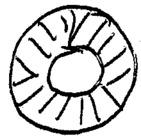
$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x \qquad x > 0$$

$$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x \qquad x > 0$$

For our problem with the vibrating membrane also

$J_1(x)$, $J_2(x)$, $J_3(x)$ ....

are of intrest, But not the singular ones. If not the drum looks like!



Back to 11.4,5 and our singular Sturm-Liouville problem

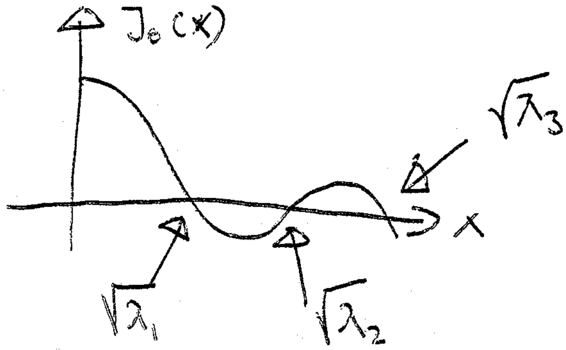
Our boundary conditions are now

1) Bounded solution at $r = 0$

2) $R(1) = 0$

$$R(r) = a \, J_0(r\sqrt{\lambda})$$

This second condition tells us which λ-values that are possible. $J_0(\sqrt{\lambda})$ must be zero



With the initial values (see 11.5)

$$u(r,0) = f(r) \quad 0 \le r \le 1$$

$$\frac{\partial u}{\partial t}(r,0) = 0 \quad 0 \le r \le 1$$

Our solution is hopefully

$$u(r,t) = \sum_{k=1}^{\infty} \underset{\text{unknown}}{c_k} \, J_0\left(\sqrt{\lambda_k}\, r\right) \cos\left(c\sqrt{\lambda_k}\, t\right)$$

$t=0$

$$f(r) = \sum_{k=1}^{\infty} c_k \, J_0\left(\sqrt{\lambda_k}\, r\right)$$

Is there a spectral theorem (like in 11.2) also here? Yes and it can be proven with help of Green function technique.

Show the ortogonality property in 5.8.14