Mathematics
DFM
Linneaus University

Partial differential equations
MA1243
fall 2010

First order PDE:s

"In contrast to ordinary differential equations, there is no unified theory of partial differential equations. Some equations have their own theories, while others have no theory at all. .....In this lecture we shall consider a case in which there is a complete theory, namely the case of one first-order equation" [1].

The simplest PDE we can think of in two dimensions ($n = 2$) is first-order, linear and homogenous.

$$a_1(x_1, x_2)\frac{\partial u}{\partial x_1} + a_2(x_1, x_2)\frac{\partial u}{\partial x_2} = 0 \tag{1}$$

Here the coefficients a_1 and a_2 are known functions and we are looking for functions $u(x_1, x_2)$ which fulfil the equation. The right way to attack this problem is to consider the following system of ODE:s.

$$\begin{aligned} x_1'(t) &= a_1(x_1, x_2) \\ x_2'(t) &= a_2(x_1, x_2) \end{aligned} \tag{2}$$

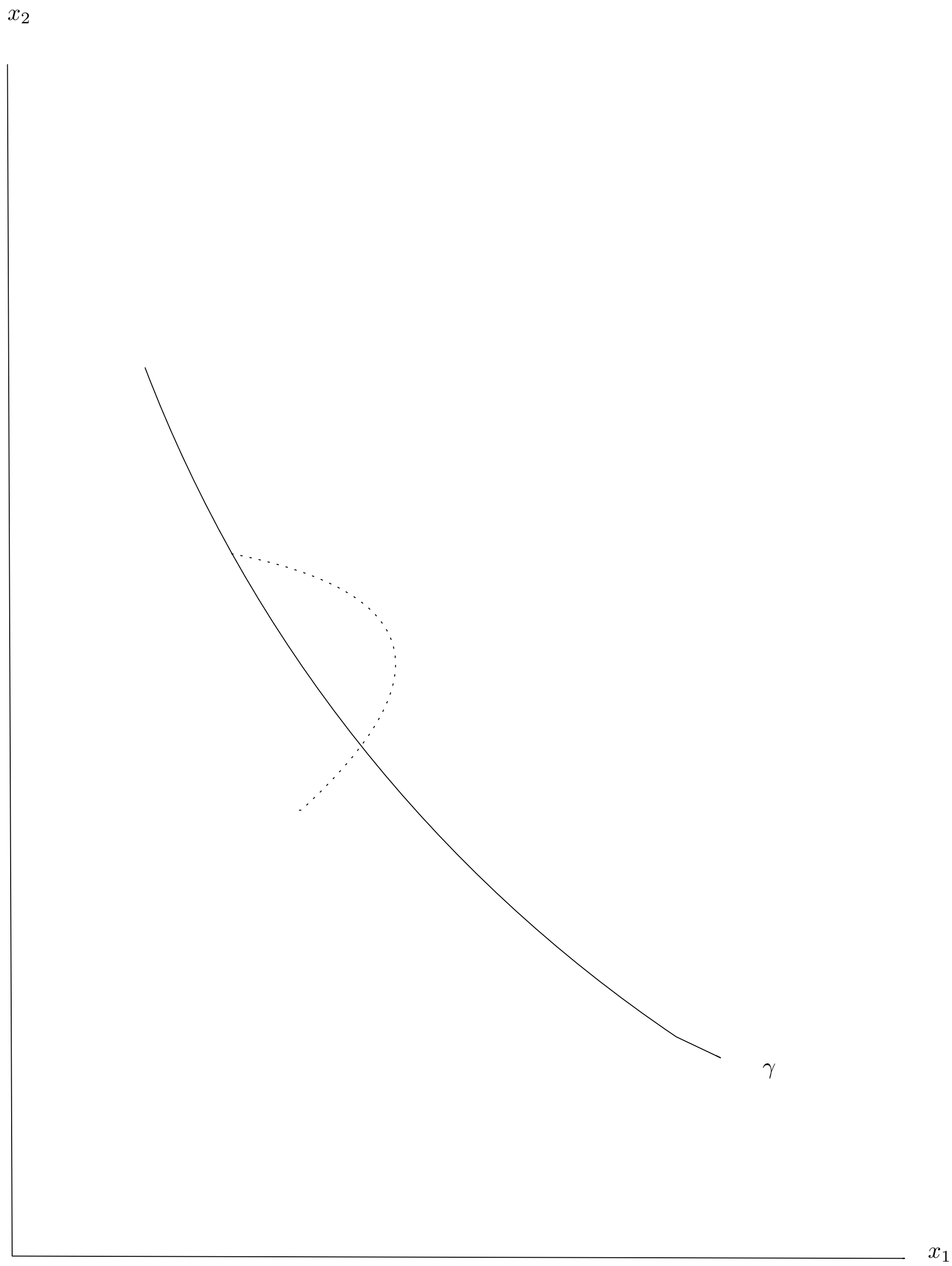
This equation is called the characteristic equation and the curves for the solutions are called characteristics. The connection between the two equations above is the following. When we are looking for constants of motion, $u(x_1, x_2)$, along the flow (2) we are searching for functions for which $\frac{du}{dt} = 0$. Using the chain rule and (2) we see that the constants of motion must fulfil (1). In a physical system an example of a constant of motion can be the kinetic plus potential energy (when friction can be neglected). So:

$u(x_1, x_2)$ is a solution to (1) $\Leftrightarrow$ $u(x_1, x_2)$ constant along the solutions to (2).

For an initial-value problem, also called Cauchy problem, is $u(x_1, x_2)$ specified along a curve γ in the plane

$$u(x_1, x_2) \mid_{\gamma} = \phi(x_1, x_2) \tag{3}$$

where $\phi(x_1, x_2)$ is called the inititial condition. The figure below illustrates that the Cauchy problem does not always have a solution for all times. (If ϕ is not constant along the full curve γ and the flow is such that a characteristic (dotted curve) cut γ also for some $t > 0$ we have a conflict.)



Ex.1 Solve

$$2\frac{\partial u}{\partial x_1} - \frac{\partial u}{\partial x_2} = 0 \tag{4}$$

when $u(x_1, 0) = e^{-x_1^2}$.

Ex.2.Solve

$$y\frac{\partial u}{\partial x} - x\frac{\partial u}{\partial y} = 0 \tag{5}$$

when $u(0, y) = \cos(y^2)$.

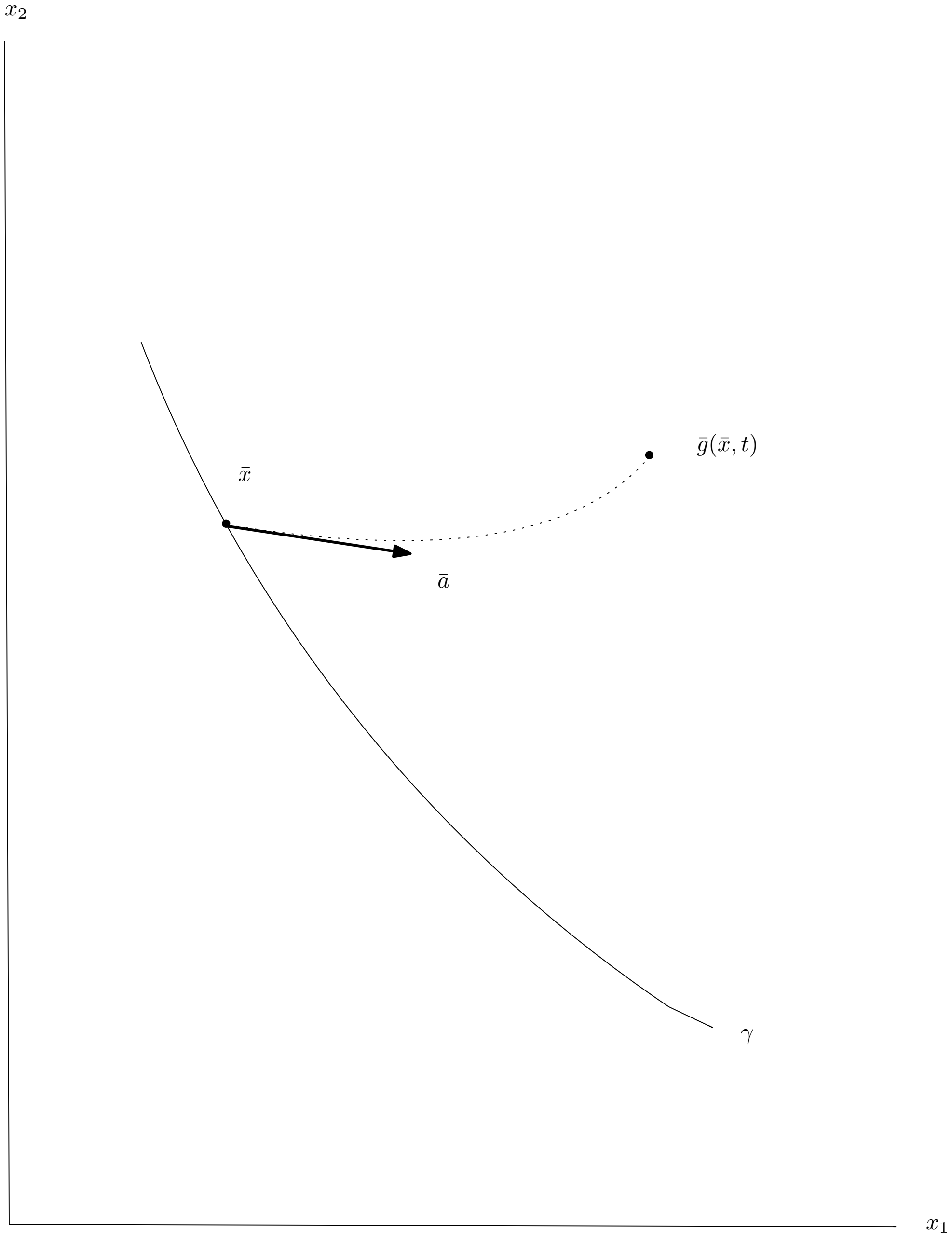
The same idea is used to solve the inhomogenous problem,

$$a_1(x_1, x_2)\frac{\partial u}{\partial x_1} + a_2(x_1, x_2)\frac{\partial u}{\partial x_2} = b(x_1, x_2) \tag{6}$$

,only difference is that now $\frac{du}{dt} = b$ along the characteristics. The solution is therefore on the form

$$u(\bar{g}(\bar{x}, t)) = \phi(\bar{x}) + \int_0^t b(\bar{g}(\bar{x}, \tau)) d\tau \tag{7}$$

where $\bar{g}(\bar{x}, t)$ is the solution to the characteristic equation with $\bar{g}(\bar{x}, 0) = \bar{x}$ on γ (see figure below).



Ex.3 Consider the equation

$$\frac{\partial u}{\partial x_1} + 2 \frac{\partial u}{\partial x_2} = 1 \tag{8}$$

when $u(0, x_2) = x_2^2$. What is $u(1, 5)$?

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References

[1] V. I. Arnold, Lectures on Partial Differential Equations, Springer , PHASIS, 2004