Inner product and selfadjoint matrices

$\mathbb{R}^2$ $\quad \bar{x}^T \bar{y} = \bar{x} \cdot \bar{y} = x_1 y_1 + x_2 y_2 = (\bar{x}, \bar{y})$

$\mathbb{R}^3$ $\quad \bar{x}^T \bar{y} = \bar{x} \cdot \bar{y} = x_1 y_1 + x_2 y_2 + x_3 y_3 = (\bar{x}, \bar{y})$

Def. of real inner product:

i) $(\bar{x}, \bar{y}) = (\bar{y}, \bar{x})$ A real number

ii) $(k\bar{x}, \bar{y}) = k(\bar{x}, \bar{y}) \quad k \in \mathbb{R}$

iii) $(\bar{x} + \bar{y}, \bar{z}) = (\bar{x}, \bar{z}) + (\bar{y}, \bar{z})$

iv) $(\bar{x}, \bar{x}) \geq 0$. Equality only if $\bar{x} = \bar{0}$

For complex vector spaces:

i') $(\bar{x}, \bar{y}) = (\bar{y}, \bar{x})^*$ complex conjug.

A complex number

$(\bar{x}, k\bar{y}) = k^* (\bar{x}, \bar{y})$

For general vector spaces, like continous functions on an interval $[\alpha, \beta]$,

$$\int_\alpha^\beta f(x)\, g(x)\, dx \quad , \quad \int_\alpha^\beta f(x)\, r(x)\, g(x)\, dx$$

$r(x)$ positive in $[\alpha, \beta]$

and $\int_\alpha^\beta f(x)\, g^*(x)\, dx$ in the complex case

fulfils the requirements above for an inner product.

For which matrix B is

$$(A\bar{x}, \bar{y}) = (\bar{x}, B\bar{y})$$ for a

given $n \times n$ matrix A?

$$\sum_{j=1}^{n} \sum_{i=1}^{n} a_{ji} x_i y_j^* = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i b_{ij}^* y_j^*$$

Answer: $b_{ij}^* = a_{ji}$

Notation A^* is used for B.
OBSERVE BOTH complex conjugation and transpose.

Ex) $A = \begin{bmatrix} i & 1-2i \\ 2i & -1 \end{bmatrix}$, $A^* = \begin{bmatrix} -i & -2i \\ 1+2i & -1 \end{bmatrix}$

If $A = A^*$ then A is <u>self adjoint</u> or <u>Hermitean</u>. $A = A^T$ (symmetric) for real matrices.

Three properties for selfadjoint matrices:

P1) If $\bar{x}$ is an eigenvector to A with eigenvalue λ then

$$\begin{array}{ccccc} (A\bar{x}, \bar{x}) & = & (\lambda\bar{x}, \bar{x}) & = & \lambda(\bar{x}, \bar{x}) \\ \| & & & & \| \\ (\bar{x}, A\bar{x}) & = & (\bar{x}, \lambda\bar{x}) & = & \lambda^*(\bar{x}, \bar{x}) \end{array}$$

so $\lambda = \lambda^*$ REAL!

P2) $A\bar{u}_1 = \lambda_1 \bar{u}_1$, $A\bar{u}_2 = \lambda_2 \bar{u}_2$ then

$$(A\bar{u}_1, \bar{u}_2) = (\lambda_1 \bar{u}_1, \bar{u}_2) = \lambda_1 (\bar{u}_1, \bar{u}_2)$$
$$=$$
$$(\bar{u}_1, A\bar{u}_2) = (\bar{u}_1, \lambda_2 \bar{u}_2) = \lambda_2 (\bar{u}_1, \bar{u}_2)$$

Real due to P1

so $\lambda_1 (\bar{u}_1, \bar{u}_2) = \lambda_2 (\bar{u}_1, \bar{u}_2)$

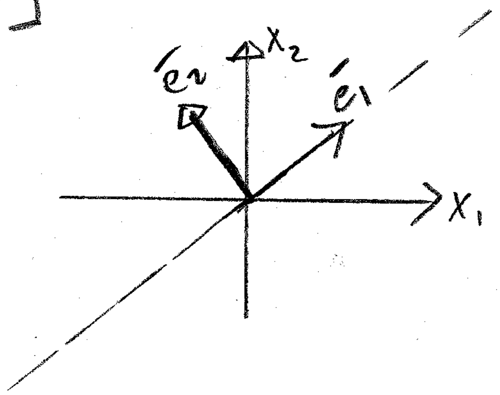
$(\bar{u}_1, \bar{u}_2) = 0$ (orthogonal)

if $\lambda_1 \neq \lambda_2$

P3) Every $n \times n$ selfadjoint matrix A has n mutually orthogonal eigenvectors

More tricky to show!

Ex) $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ Reflexion in $x_1 = x_2$ line



$\bar{e}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\lambda_1 = 1$

$\bar{e}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ $\lambda_2 = -1$

$$\bar{e}_i \cdot \bar{e}_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$