Mathematics
DFM
Linneaus University

Partial differential equations
4MA103
fall 2012

Finding eigenvalues numerically

1. Find numerically the two lowest eigenvalues to the problem

$$-y'' - \frac{5}{1+x^2}y = \lambda y.$$

when $y(\pi) = y(2\pi) = 0$. This problem is on Liouville normal form. Lecture 12 is about the procedure to obtain this form for the Sturm-Liouville problems.

Start by solving the ODE for $\lambda = 4$ and initial values $y(\pi) = 0, y'(\pi) = 1$. Note, now you have to use NDSolve. Plot the solution from π to 2π. Counting the number of zeros of $y(x)$ *inside* the interval you can now tell how many eigenvalues there are below $\lambda = 4$ (see lecture 9). It also illuminating to make a parametric plot (use ParametricPlot) in the (y', y)-plane. $\pi \leq x \leq 2\pi$ is the parameter and you start in (1,0). Can a curve like this one go through the origin (0,0)?

A quick estimate of the two eigenvalues can now be obtained by repeating the plots for some lower λ values and observing what happens at the right end. When $y(2\pi) = 0$ you have found an eigenvalue!

2. Section 11.3 is about non-homogeneous boundary value problems. Consider the problem 11.3.2

$$-y'' = 2y + x.$$

$y(0) = y'(1) = 0$. Find the exact solution by paper and pen and by DSolve. Usually we have to live with series solutions. Find eigenfunctions $y_n(x)$ and eigenvalues λ_n to $-y''$ and the given boundary conditions. Normalize the eigenfunctions $y_n(x)$ so

$$\int_0^1 y_n(x)^2 dx = 1.$$

Now it is time to expand the inhomogeneous term x in the basis $y_n(x)$,

$$x = \sum_{n=1}^{\infty} c_n y_n(x).$$

The Fourier coefficient c_j is obtained by multiplying both sides with $y_j(x)$ and integrating from 0 to 1. The series expansion for the solution to the problem is

$$y(x) = \sum_{n=1}^{\infty} b_n y_n(x)$$

and the coefficients are $b_n = c_n/(\lambda_n - 2)$. See section 11.3 for the details. In practice you have to truncate the infinte sum at some $n = N$. Try $N = 10, 20$ and 30 and compare with exact solution.