Mathematics
DFM
Linneaus University

Partial differential equations
4MA103
fall 2012

Green function, Liouville's normal form

1. Plot the solution to the initial value problem

$$y'' + \frac{e^x + e^{-x}}{2} y = 0.$$

when $y(0) = 0, y'(0) = 1$. But before that, think about how the solution will look like. When an eigenvalue problem is on Liouville's normal form (see lecture 12)

$$-y'' + q(x)y = \lambda y.$$

we can think of $\lambda - q(x)$ as an x dependent angular frequency squared, $\omega^2(x) = \lambda - q(x)$, when λ is sufficiently large.

2. Continuation of the problem 11.3.2 from last week.

$$-y'' = 2y + x.$$

$y(0) = y'(1) = 0$.
The Green function for this problem is

$$G(x, s) = (cos\sqrt{2}s \cdot sin\sqrt{2}x + tan\sqrt{2} \cdot sin\sqrt{2}s \cdot sin\sqrt{2}x)/\sqrt{2}$$

for $x \leq s$ and

$$G(x, s) = (cos\sqrt{2}x \cdot sin\sqrt{2}s + tan\sqrt{2} \cdot sin\sqrt{2}s \cdot sin\sqrt{2}x)/\sqrt{2}$$

for $x \geq s$. Use G to calculate $y(x)$. $y(x) = \int_0^1 G(x, s)\ s\ ds$. Compare with exact solution. Remember you have to split the integral in two parts, one from 0 to x and another from x to 1. Use command Integrate. Plot also the Green function for a fixed s, say $s = 1/2$, and observe the corner at $x = s$.

3. The Green's function is designed to give the value of the non-homogeneous term $f(x)$ at point x. Therefore acting with the differential operator on G gives a spike (Dirac's delta-function) located at $s = x$. A unit step located at $x = 0$ can be obtained by taking the limit $\epsilon \to 0^+$ of the following function

$$h(x, \epsilon) = \frac{1}{\pi} arctan \frac{x}{\epsilon}$$

Plot the function for some fixed small ϵ in a region around the origin. The derivative of a step is a spike! So plot also

$$h'(x, \epsilon) = \frac{1}{\pi} \frac{\epsilon}{\epsilon^2 + x^2}$$

for some fixed small ϵ in a region around the origin. What is the area under the curves? Finally, calculate

$$\int_{\infty}^{\infty} \frac{1}{\pi} \frac{\epsilon f(x)}{\epsilon^2 + x^2} \, dx$$

for some function f and check that the value comes closer and closer to $f(0)$ when ϵ decreases. Maybe you have to use NItegrate and truncate the integral.