How to obtain
Liouville's normal form

The method is shown by an example

$$-x\,y''(x) - 2x\,y'(x) = \lambda\, y(x) \qquad x>0$$

I) Multiply with a function $r(x)$ to obtain self-adjoint form.

condition $\quad \big(r(x)\,x\big)' = 2x\,r(x)$

$$r(x) = \frac{e^{2x}}{x} \qquad \text{Show!}$$

Now: $-\left(e^{2x}\,y'(x)\right)' = \lambda\, r(x)\, y(x)$

II) Introduce new variabel

$Z = \int_{\alpha}^{x} \sqrt{\frac{r}{p}}\, dx = 2\sqrt{x}$ in our case

and new function $v(z(x)) = y(x)$

then

$$y'(x) = v'(z)\,\frac{1}{\sqrt{x}}$$

$$y''(x) = v''(z)\,\frac{1}{(\sqrt{x})^2} + v'(z)\cdot\frac{-1}{2x^{3/2}}$$

Show that the equation for $V(z)$ is

$$-V''(z) - V'(z)\left[z - \frac{1}{z}\right] = \lambda V(z)$$

We got rid of $r(x)$

II Eliminate first-order term by

$$V(z) = g(z)\,U(z)$$

$g(z)$ is chosen such that U'-term disappears

Show that $g(z) = \sqrt{z}\, e^{-z^2/4}$

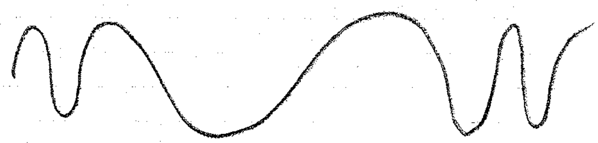
At last Liouville's normal form

$$\boxed{-U''(z) + \left(\frac{3}{4z^2} + \frac{z^2}{4}\right) U(z) = \lambda U(z)}$$

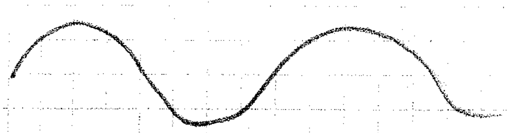
$P \equiv 1$ OBSERVE $U''(z) = -\omega^2 U$

$r \equiv 1$ has solutions $\cos\omega z$, $\sin\omega z$

Here $\omega^2(z)$



instead of



This is a convenient form when proving part 3 of the spectral theorem.

$$\lambda_1 < \lambda_2 < \lambda_3 < \dots\dots$$

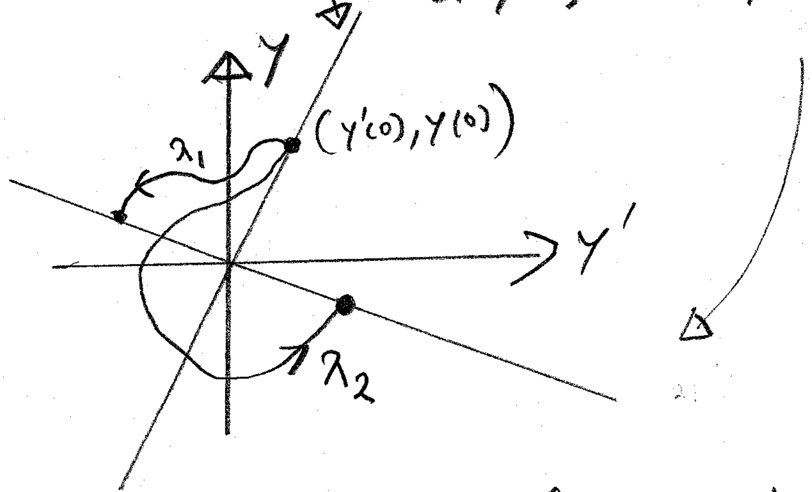
$$\lambda_n \to \infty \quad \text{when } n \to \infty.$$

$$-y'' + q(x)y = \lambda y, \qquad b_{11}\,y(0) + b_{12}\,y'(0) = 0$$

$$b_{21}\,y(1) + b_{22}\,y'(1) = 0$$

$$y'' + (\lambda - q)y = 0$$

Larger $\lambda \to$
Larger "Frequency"



Fine tuning of λ to fit the BC at the right end point!

ϕ_1 has no zeros (nodes) in $]0,1[$

ϕ_2 has 1 zero (node) in $]0,1[$

etc.