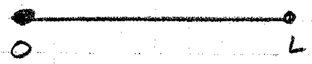
11.1 and 11.2

10/11 and 12/11

Separation of variables in a PDE gives rise to problems of the type

$$X''(x) + \lambda X(x) = 0 \qquad (*)$$
$$X(0) = X(L) = 0$$



$0 < x < L$

Three generalizations of $(*)$

i) $$(p(x) X')' - q(x) X = -\lambda r(x) X \qquad (**)$$

ii) $$\frac{\partial u}{\partial x}(0,t) - h_1 u(0,t) = 0$$

$$\frac{\partial u}{\partial x}(L,t) + h_2 u(L,t) = 0$$

i) + ii) are called Sturm-Liouville problems

iii) more complicated geometries
see 11.5

Note that $X(x) \equiv 0$ is a trivial solution to (**).

For some special values of λ (eigenvalues) there are non-trivial solutions (eigenfunctions). They can be used for series expansions.

Exactly as for the Fourier series. For example

$$X'' = -\lambda X$$
$$X(0) = X(\pi) = 0$$
$$X_n = c_n \sin nx$$

11.1.11 $\quad X \to Y(x)$

Why $py'' + p'y'$ as in (**)?

$$P(x)y''(x) + Q(x)y'(x) + R(x)y = 0$$

Multiply with $\mu(x)$

$$\mu P y'' + \mu Q y' + \mu R y = 0$$

We want $[\mu P y']' + \mu R y = 0$

$$\Leftrightarrow$$

$$\mu P y'' + y'(\mu P)' + \mu R y = 0$$

Works if

$$\mu' P + \mu P' = Q\mu$$

$$\mu' P = \mu (Q - P')$$

$P \neq 0$
in the interval

$$\frac{\mu'}{\mu} = \frac{Q}{P} - \frac{P'}{P}$$

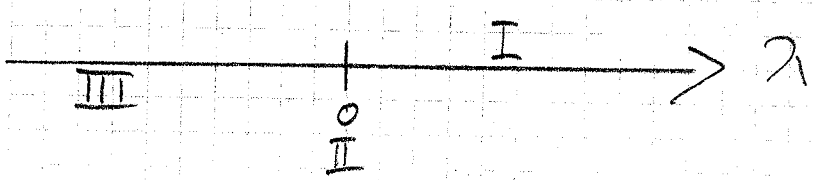
$$\ln|\mu| = \int_{x_0}^{x} \frac{Q}{P}\, ds - \ln|P|$$

$$\mu(x) = \frac{1}{P(x)} \exp\left(\int_{x_0}^{x} \frac{Q(s)}{P(s)} \right)$$

Do II, 1. 13

II. 1. 8)

$$y''(x) + \lambda y(x) = 0$$
$$y'(0) = 0$$
$$y(1) + y'(1) = 0$$



(I) $\lambda = m^2$

$$y'' = -m^2 y$$
$$y = A\cos mx + B \sin mx$$
$$y'(x) = -Am \sin mx + Bm \cos mx$$

$$y'(0) = -Am \sin m0 + Bm \cos m0$$

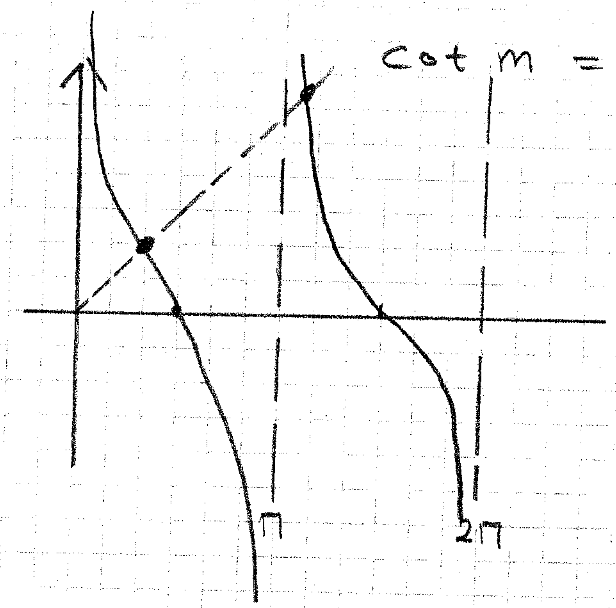
$$y(1) + y'(1) = A(\cos m - m \sin m) + B(\sin m + m \cos m) = 0$$

$$\underbrace{\begin{bmatrix} 0 & m \\ \cos m - m \sin m & \sin m + m \cos m \end{bmatrix}}_{M} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\det M = 0$ for non-trivial solutions

$$\det M = -m(\cos m - m \sin m) = 0$$

$$\cot m = m$$



$$\lambda_m \approx (m-1)^2 \pi^2$$

Do $\lambda = 0$

and

$\lambda = -m^2$

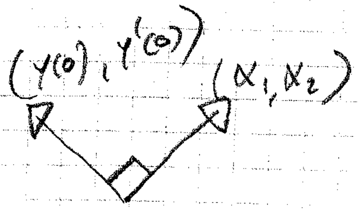
11.2 Sturm-Liouville boundary value problems

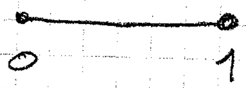
(1) $\quad [p(x)\, y']' - q(x)\, y + \lambda r(x)\, y = 0$

Separated boundary values:

$$\alpha_1\, y(0) + \alpha_2\, y'(0) = 0$$

$$\beta_1\, y(1) + \beta_2\, y'(1) = 0$$





p, p', q, r continous

$p(x) > 0$

$r(x) > 0$ $\quad 0 \le x \le 1$

(1) can be written as

$$\mathcal{L}y = L[y] = \lambda r(x)\, y(x)$$

$$\mathcal{L}y = -(py')' + qy$$

$$\mathcal{L}y = \lambda r(x)\, y$$

Lagrange identity:

$$\int_0^1 \mathcal{L}u(x)\, v(x)\, dx = \int_0^1 -(pu')'v + quv\, dx$$

$= \ldots\ldots$ 2 partial integrations $=$

$$= \int_0^1 u \mathcal{L} v \, dx + \Big(\underbrace{\big[-p [u'v - uv'] \big]_0^1}_{\text{boundary term}} \Big)$$

If u and v fulfils the boundary conditions then this term is zero.

$(u(1), u'(1)) \,/\!/\, (v(1), v'(1))$

etc.

$\mathcal{L}$ is then said to be symmetric or self-adjoint.

(Hermitean in quantum mechanics)

Observe it is a property of <u>both</u> operator and boundary conditions

Do 11.2.15