Introduce the notation

$$(u, v) = \int_0^1 u(x)\, v(x)\, dx$$ scalar-product

Then

$$\big(L[u], v \big) = \big(u, L[v] \big)$$ for

self-adjoint operators.

In quantum mechanics the functions can be complex and also the operators, for example $\hat{p} = -i\hbar \frac{\partial}{\partial x}$.

The scalar product is then modified in the following way

$$(u, v) = \int_0^1 u(x)\, \overline{v}(x)\, dx$$

complex conjugation

I prefere v^* etc

Theorem 11.2.1

All the eigenvalues of the SL-problem are real.

Proof: $(L[\phi], \phi) = \lambda (r\phi, \phi)$

for an eigenfunction ϕ.

But we also have

$(L[\phi], \phi) = (\phi, L[\phi])$

$= \lambda^*(\phi, r\phi)$ for SL-problems

Since $r(x)$ is real

$$(\phi, r\phi) = \int_0^1 r(x)\, \phi(x)\, \phi^*(x)\, dx$$

which is positive

Then $\lambda = \lambda^*$

<u>Th 11.2.2</u> Eigenfunctions $\phi_1(x), \phi_2(x)$ belonging to <u>different</u> eigenvalues are orthogonal, that is

$$\int_0^1 r(x)\, \phi_1(x)\, \phi_2^*(x)\, dx = 0$$

Proof: $\mathcal{L}\phi_1 = \lambda_1 \phi_1$

$r \equiv 1$ here $\mathcal{L}\phi_2 = \lambda_2 \phi_2$

$\underline{\lambda_1(\phi_1, \phi_2)} = (L[\phi_1], \phi_2) = (\phi_1, L[\phi_2]) =$

$= \underline{\lambda_2(\phi_1, \phi_2)}$

So $(\phi_1, \phi_2) = 0$ since $\lambda_1 \neq \lambda_2$

Theorem 11.2.3

To each eigenvalue there is exactly one eigenfunction (no degeneration). Moreover

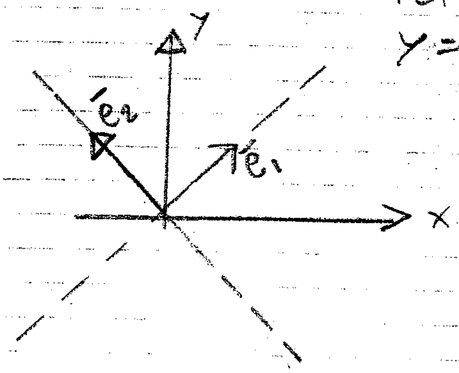
$$\lambda_1 < \lambda_2 < \lambda_3 < \dots \lambda_n < \dots$$

$$\lambda_n \to \infty \quad \text{when} \quad n \to \infty$$

11.2 continued

Linear algebra:

reflection in $y = x$



$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$A\bar{x} = \lambda \bar{x}$$

$\bar{e}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ with eigenvalue $+1$

$\bar{e}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ with eigenvalue -1

$$\bar{e}_i^t \cdot \bar{e}_2 = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

$$\bar{v} = c_1 \bar{e}_1 + c_2 \bar{e}_2 \qquad c_i = \bar{v}^t \bar{e}_i = \bar{v} \cdot \bar{e}_i$$

$\bar{y} = A\bar{x}$ has a unique solution if not the homogenous equation $A\bar{x} = \bar{0}$ has a non-trivial solution. The solution is then $\bar{x} = A^{-1}\bar{y}$

Sturm - Liouville theory

$$L[y] = -(p y)' + q y = \lambda r(x) y(x)$$

$$\alpha_1 y(0) + \alpha_2 y'(0) = 0$$
$$\beta_1 y(1) + \beta_2 y'(1) = 0$$

Infinitely many real eigenvalues

$$\lambda_1 < \lambda_2 < \lambda_3 < \ldots\ldots$$

The eigenvectors $\phi_i(x)$ $i = 1, 2, 3 \ldots$ are **orthogonal** $(\phi_i(x), r(x)\,\phi_j(x)) = \delta_{ij}$

If a continous function $f(x)$ fulfils the boundary condition then $f(x) = \sum_{k=1}^{\infty} c_k\,\phi_k(x)$ uniformly

$c_k = (f(x), r(x)\,\phi_k(x))$. (Th 11.2.4)

see also 11.6

In 11.3 we will see that the non-homogenous problem

$$L[y] = \mu\, r(x)\, y(x) + f(x)$$

, with separated b.c. as before, has an unique solution if not the homogenous problem $L[y] = \mu\, r(x)\, y$ has a non-trivial solution.

Generalized S-L theory:

i) Higher derivatives
$n = 2, 4, 6, \ldots$ must be even
since integrating by parts
odd times give minus sign
in front of $y^{(n)}(x)$

ii) Other boundary conditions.
Periodic conditions give rise
to degeneracies

iii) Singular S-L problems

relaxed conditions
on $p(x)$, $q(x)$, $r(x)$
see 11.4 and chapter 5

infinite
interval