Green funktioner

From linear algebra we know that

$$\vec{x} = (A - \mu I)^{-1} \vec{b}$$

is a solution to

$$A\vec{x} - \mu\vec{x} = \vec{b}$$

if the inverse matrix exists.

Is there something similar for our non-homogeneous problems in 11.3 ?

$$\mathcal{L} y = f$$

It is tempting to write the solution as

$$y = \mathcal{L}^{-1} f$$

but what is the meaning of $\mathcal{L}^{-1}$? .... Maybe integration!!

This leads us to the Greens function (see pages 692-694)

11.3.28 $$-y''(x) = f(x) \quad \left(\begin{matrix} f \text{ is} \\ \text{known}\end{matrix}\right)$$

has solution

$$y = \phi(x) = c_1 + c_2 x - \int_0^x (x-s) f(s)\, ds. \quad (*)$$

see section 3.7. You can also take the derivatives of $(*)$

$$\phi'(x) = c_2 - (x-x) f(x) - \int_0^x f(s)\, ds$$

$$= c_2 - \int_0^x f(s)\, ds$$

$$\phi''(x) = -f(x)$$

(see appendix how to take the derivative of the integral above)

If we have the boundary conditions

$$y(0) = y(1) = 0$$

we get

$$y(0) = c_1 = 0$$

$$y(1) = c_2 - \int_0^1 f(s)(1-s)\, ds = 0$$

So

$$y(x) = \int_0^1 x(1-s) f(s)\, ds - \int_0^x (x-s) f(s)\, ds$$

$$= \int_0^x s(1-x) f(s)\, ds + \int_x^1 x(1-s) f(s)\, ds$$

$$= \int_0^1 G(x,s) f(s)\, ds \qquad (**)$$

where the Green's function is

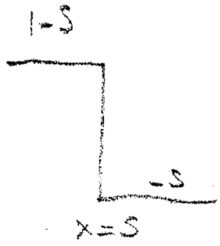
$$G(x,s) = \begin{cases} s(1-x) & 0 \le s \le x \\ x(1-s) & x \le s \le 1 \end{cases}$$

The operator in (**) is an integral operator acting on f. Can be regarded as the inverse to $\mathcal{L}$.

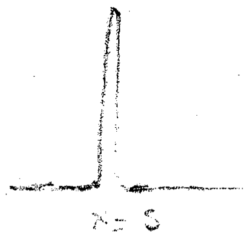
The nice thing is that the Green's function does not depend on $f(s)$.

Ex) Solve $y'' = -1$ $\quad y(0) = y(1) = 0$ in the ordinary way and with help of (**)

Taking $\frac{dG}{dx}$ for fixed s we get



so it makes a jump of one unit at $x = s$. Taking another derivative $\frac{d^2G}{dx^2}$ is not allowed but allowing for little smoothness we get



This spike is designed to pick out the value of f at $s = x$

FORMALLY: $$y''(x) = \int_0^1 \frac{d^2G}{dx^2} f(s)\, ds = -f(x)$$

Example: $$h_\varepsilon(x) = \frac{1}{\pi} \arctan \frac{x}{\varepsilon}$$

$$h_\varepsilon'(x) = \frac{1}{\pi} \frac{\varepsilon}{\varepsilon^2 + x^2}$$

Draw the curves and let $\varepsilon \to 0$

Area under h_ε' is 1.

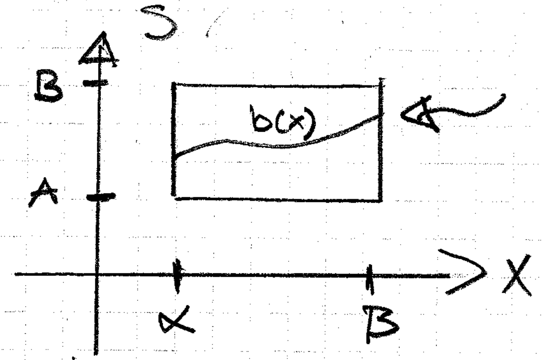
$$\int h_\varepsilon'(x) f(x)\, dx \approx f(0)$$

Appendix

FROM Persson & Böiers
Analys i flera variabler

$$F(x) = \int_{x=a}^{b(x)} f(x,s)\,ds \qquad \text{then}$$

$$F'(x) = \int_{a}^{b(x)} \frac{\partial f}{\partial x}(x,s)\,ds + f(x, b(x))\, b'(x)$$



f and $\frac{\partial f}{\partial x}$ continous here

$b(x)$ a C^1-function.

$A < a < B$.

Proof: Introduce $G(x,t) = \int_a^t f(x,s)\,ds$

so $F(x) = G(x, b(x))$

$$G_x'(x,t) = \int_a^t \frac{\partial f}{\partial x}(x,s)\,ds$$

Use mean-value theorem!

$$G_t'(x,t) = f(x,t)$$

Chain Rule gives

$$F'(x) = G_x'(x, b(x)) + G_t'(x, b(x))\, b'(x)$$

$$= \int_a^{b(x)} \frac{\partial f}{\partial x}(x,s)\,ds + f(x, b(x)) \cdot b'(x).$$

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