# 5. Series solutions of Second Order Linear equations

5.1 $x_0 = 0$ Power series

Some Taylorseries near $x_0 = 0$

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$$ Converges absolutely when $|x|<1$

Diverges at $x = \pm 1$

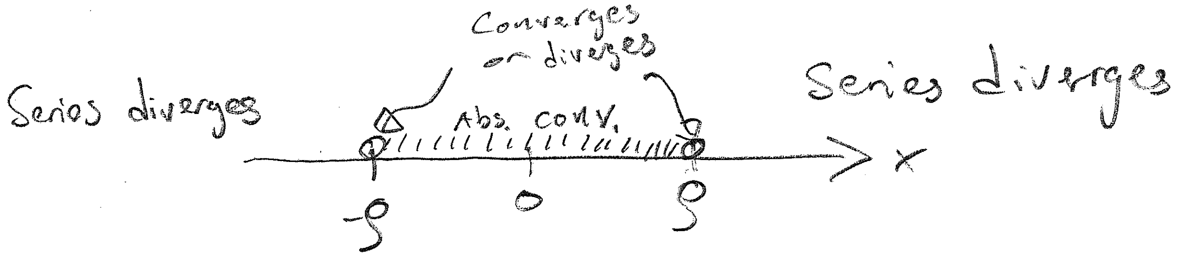
$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \dots = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k}$$

Converges absolutely for $|x|<1$

Diverges for $x = -1$ and

Converges for $x = 1$.

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} \dots \approx 0.69$$




ρ - radius of convergence

Ratiotest: $$\frac{|a_{k+1} x^{k+1}|}{|a_k x^k|} \to \underbrace{\left|\frac{a_{k+1}}{a_k}\right| |x|}_{= L|x|} \quad \text{when } k \to \infty$$

The power series converges if $L|x|<1$, diverges if $L|x|>1$.

For $L|x|=1$ the test is inconclusive

Ex) $\left| \frac{(-1)^{k+2}}{(-1)^{k+1}} \frac{x^{k+1}}{k+1} \cdot \frac{k}{x^k} \right| \to |x|$ when

$k \to \infty$ so radius of convergence is $|x|<1$

Ex) $\sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x$, $\left| \frac{x^{k+1}}{(k+1)!} \frac{k!}{x^k} \right| \to 0 \cdot |x|$

when $k \to \infty$. $L=0 \Rightarrow \rho=\infty$,

The series converges for all $x$

Inside the radius of convergence the power series can be treated as expected from finite sums.

For example termwise derivation is allowed.

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$$

$$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \ldots$$

$$f''(x) = 2a_2 + 6a_3 x + \ldots$$

$a_0 = f(0)$, $a_1 = f'(0)$, $a_2 = f''(0)/2!$

 $a_3 = f^{(3)}(0)/3!$. That is the Taylor series

5.1.2 $\sum_{n=0}^{\infty} \frac{n}{2^n} x^n = \frac{x}{2} + \frac{x^2 \cdot 2}{4} + \frac{3x^3}{8} +$

Determine $\rho$

Extra: Can you express the series by elementary functions? Hint $f(x) = 1 + \frac{x}{2} + \frac{x^2}{4} + \frac{x^3}{8} + \ldots = \frac{1}{1-\frac{x}{2}}$

5.2) $P(x)y''(x) + Q(x)y'(x) + R(x)y = 0$

Ex) $x^2 y''(x) + x y'(x) + (x^2 - \nu^2) y = 0$

Bessel eq.

$(1-x^2) y'' - 2x y' + \alpha(\alpha+1) y = 0$

Legendre eq.

P, Q, R most often polynomials

We seek a power series solution near a point $x_0$

$$P(x_0) \begin{cases} = 0 & x_0 \text{ ordinary point} \\ \neq 0 & x_0 \text{ ordinary point} \end{cases}$$

Near an ordinary point a unique solution exists (see ODE-course, Theorem 3.2.2)

Solve

5.2.7 $1 \cdot y''(x) + x \cdot y'(x) + 2y = 0$ (*)

near $x_0 = 0$

$P(x) = 1$, $Q(x) = x$, $R(x) = 2$

$x_0 = 0$ an ordinary point

Ansatz $y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots$

We assume it converges near $x_0 = 0$.

$y'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \ldots = \sum_{n=0}^{\infty} a_n n x^{n-1}$

$y''(x) = 2a_2 + 6a_3 x + \ldots =$

$\sum_{n=0}^{\infty} a_n n(n-1) x^{n-2}$

$x y'(x) = \sum_{n=0}^{\infty} a_n n x^n$

All this put together in (*)

$\sum_{n=0}^{\infty} a_n n(n-1) x^{n-2} + a_n n x^n + 2a_n x^n = 0$

Collect terms with the same degree

$x^0: \quad a_2 \cdot 2 \cdot 1 + 2a_0 = 0, \quad a_2 = -a_0$

$x^1: \quad a_3 \cdot 3 \cdot 2 + a_1 \cdot 1 + 2a_1 = 0, \quad a_3 = \frac{-a_1}{2}$

$x^2: \quad a_4 \cdot 4 \cdot 3 + a_2 2 + 2a_2 = 0, \quad a_4 = \frac{-a_2}{3} = \frac{a_0}{3}$

Two constants, $a_0$ and $a_1$, will be determined by initial conditions $y(0)$ and $y'(0)$.

$x^n: \quad a_{n+2}(n+2)(n+1) + a_n n + 2a_n = 0$

$$\boxed{a_{n+2} = \frac{-a_n}{(n+1)}}$$

$$y_1(x) = a_0\left(1 - x^2 + \frac{x^4}{3} - \frac{x^6}{15} + \dots\right)$$

$$= a_0 \sum_{k=0}^{\infty} x^{2k} \cdot \frac{(-1)^k}{(2k-1)!!} \qquad (-1)!! = 1$$

$$y_2(x) = a_1\left(x - \frac{x^3}{2} + \frac{x^5}{8} - \frac{x^7}{48} + \dots\right) = a_1 x e^{-x^2/2}$$

5.3) $$P(x)\,y''(x) + Q(x)\,y'(x) + R(x)\,y = 0 \qquad (*)$$

$$p(x) = Q(x)/P(x)$$

$$q(x) = R(x)/P(x)$$

At ordinary points $x_0$:

$$p(x) = p_0 + p_1(x-x_0) + p_2(x-x_0)^2 + \dots$$

$$q(x) = q_0 + q_1(x-x_0) + q_2(x-x_0)^2 + \dots$$

Th 5.3.1 If $x_0$ is an ordinary point the series solutions $y_1$ and $y_2$ to $(*)$ are analytic at $x_0$.
$\rho$ for these series is $\geq \min\{\rho \text{ for } p, \rho \text{ for } q\}$