Problems 30/11 and 7/12

1.1.11 and 13. We want to multiply

$$P(x)\,y''(x) + Q(x)\,y'(x) + R(x)\,y = 0$$

with an integrating factor $\mu(x)$ to get the desired form

$$[\mu(x)\,P(x)\,y']' + \mu(x)\,R(x)\,y = 0$$

$\mu(x)P(x) = p(x)$
$\mu(x)R(x) = q(x)$ in 11.2. Show that μ is a solution of the ODE

$$P\mu' = (Q - P')\mu$$

Show that $\mu(x) = \dfrac{1}{P(x)}\, e^{\int_{x_0}^{x} \frac{Q(s)}{P(s)}\,ds}$

Apply this method to the Bessel equation

$$x^2 y''(x) + x\,y'(x) + (x^2 - \nu^2)\,y(x) = 0$$
$$\nu \in \mathbb{R}$$

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2) $-y''(x) = \lambda y(x)$, $y(0) = 0$, $y'(L) = 0$

Show that $\int_0^L \phi_m(x)\,\phi_n(x)\,dx = 0$ if $n \neq m$

Here ϕ_m and ϕ_n are eigenfunctions that corresponds to different eigenvalues $\lambda_m \neq \lambda_n$. Hint: multiply

$$\phi_m'' + \lambda_m \phi_m = 0 \quad \text{with } \phi_n(x)$$

and

$$\phi_n'' + \lambda_n \phi_n = 0 \quad \text{with } \phi_m(x)$$

and integrate by parts.

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3) Is the operator

$$(1+x^2)y''(x) + 2x\,y'(x) + y = 0$$

$$y'(0) = 0, \quad y(1) + 2y'(1) = 0$$

self-adjoint?

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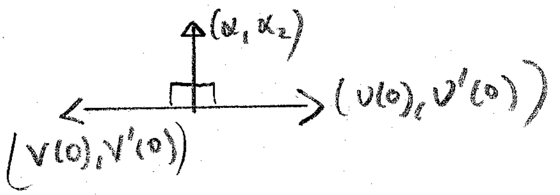
4) The eigenvalues to SL-problems are real but can the eigenfunctions be complex? Assume $\phi(x) = U(x) + i\,V(x)$, show that U and V also are eigenfunctions (ϕ is an eigenfunction). Can they be linearly independent? Calculate the Wronskian

$$\begin{vmatrix} U(0) & V(0) \\ U'(0) & V'(0) \end{vmatrix}$$



Conclusion?

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5) A generalized SL-problem is

$$-y''(x) = \lambda y(x) \quad \text{in } [-\pi, \pi]$$

with BC

$$y(-\pi) = y(\pi)$$

$$y'(-\pi) = y'(\pi)$$

OBSERVE! Not separated

Show that degeneration appears!

That is more than one eigenfunction to each eigenvalue.

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6) Solve nonhomogenous B.V.P.

$$-y''(x) = 2y(x) + x$$
$$y(0) = 0$$
$$y(1) = 0$$

using eigenfunction expansion to the operator $-y''(x)$. Compare with standard solution $y(x) = y_h(x) + y_p(x)$

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7) (singular SL-problem) The equation $-(1-x^2)y''(x) + x\,y'(x) = \lambda y$ is Chebyshev's equation

a) Rewrite it as

$$-\left[\sqrt{1-x^2}\, y'\right]' = \frac{\lambda y}{\sqrt{1-x^2}}, \quad -1 < x < 1$$

b) Show that the boundary value problem with B.C.

y, y' bounded when $x \to 1$ and $x \to -1$

is self-adjoint

c) $\lambda_0 = 0 \quad y_0(x) = 1$
$\lambda_1 = 1 \quad y_1(x) = x$
$\lambda_2 = 4 \quad y_2(x) = 1 - 2x^2$
$\lambda_n = n^2$

$$\int_{-1}^{1} \frac{y_m(x)\, y_n(x)}{\sqrt{1-x^2}}\, dx = 0 \quad m \neq n$$

Show in a particular case