Convergence of series expansions in eigenfunctions to $-y''(x) + q(x)y(x) = \lambda y(x)$ and separated B.C.

$C[0,1]$ denotes the space of continous functions on the interval $[0,1]$.

Two norms on that space:

1) $$|u| = \sup_{0 \le x \le 1} |u(x)|$$

2) $$\|u\| = \left(\int_0^1 |u(x)|^2 \, dx \right)^{1/2}$$

Show that $\|u\| \le |u|$

A sequence of functions $(u_N)_{N=1}^{\infty}$ converges <u>uniformly</u> towards a function $u \in C[0,1]$ if

$$|u_N - u| \to 0 \quad \text{when} \quad N \to \infty$$

If $\|u_N - u\| \to 0$ when $N \to \infty$

we have <u>convergence in L^2-norm</u>, or in the <u>mean square sense</u>

With <u>pointwise convergence</u> we mean

$$\lim_{N \to \infty} u_N(x) = u(x) \quad \text{for all } x \in [0,1]$$

see 11.6.4

uniform convergence $\Longrightarrow$ pointwise convergence

Let's now be more precise than in Th 11.2.4.

The question is for which $y(x)$

$$y(x) = \sum_{k=1}^{\infty} c_k \, Y_k(x) \quad \text{converges ?}$$

$$c_k = \int_0^1 y(x) \, Y_k(x) \, dx = (y(x), Y_k(x))$$

First assume $y(x)$ fulfils the boundary conditions and is as smooth as the Y_k's. Then we have uniform convergence.*

I will (almost!) show L^2-convergence below.

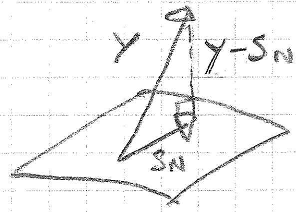
If we are satisfied with mean square convergence we can relax the constraints on $y(x)$. $y(x) \in C[0,1]$ is for example ok for L^2-convergence.

*) To prove this the Green's function is used.

Proof of L^2-convergence of $y(x) = \sum_{k=1}^{\infty} c_k Y_k(x)$

$c_k = (y, Y_k)$, $y(x)$ fulfils the boundary conditions, is continous up to n:th derivative, n - order of operator.

1) Bessel's inequality



$$\|y\|^2 \geq \sum_{k=1}^{\infty} (y, Y_k)^2$$

Proof: Put $d_k = (y, Y_k)$ and

$$S_N(x) = \sum_{k=1}^{N} d_k Y_k$$

$$\|y - S_N\|^2 = \left(y - \sum_{k=1}^{N} d_k Y_k ,\ y - \sum_{k=1}^{N} d_k Y_k \right) =$$

$$\|y\|^2 - \sum_{k=1}^{N} d_k^2$$

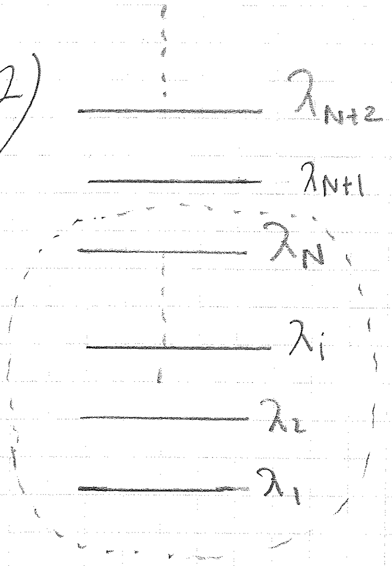
Thus $\|y\|^2 = \sum_{k=1}^{N} d_k^2 + \|y - S_N\|^2 \geq \sum_{k=1}^{N} d_k^2$

Equality if and only if $\|y - S_N\| \to 0$ when $N \to \infty$.

In step 2 we will also show that $\|y\|^2 \leq \sum_{k=1}^{\infty} (y, Y_k)^2$.

Then we know that we have equality in Bessel's inequality and therefore L^2-convergence, $\|y - S_N\| \to 0$, when $N \to \infty$

2)



$y - S_N$ orthogonal to $y_1, y_2, y_3, \ldots\ldots y_N$

$(y - S_N, y_i) = d_i - d_i = 0, \; 1 \leq i \leq N$

Then one can show

$$\left(L(y - S_N), y - S_N \right) \geq \lambda_{N+1} \left(y - S_N, y - S_N \right)$$

L is the differential operator.

Rewriting this inequality gives

$$\left((y, y) - \sum_{k=1}^{N} d_k^2 \right) \leq \frac{1}{\lambda_{N+1}} \left((Ly, y) - \sum_{k=1}^{N} d_k^2 \lambda_k \right)$$

Let now $N \to \infty$

Remember $\lambda_{N+1} \to +\infty$ when $N \to \infty$.

Replace $\sum_{k=1}^{N} d_k^2 \lambda_k \to \sum_{k=1}^{N'} d_k^2 \lambda_k$, a finite sum over negative eigenvalues

Finally $(y, y) - \sum_{k=1}^{\infty} d_k^2 \leq 0$

Together with Bessel's inequality we know that $\| y - S_N \| \to 0$ when $N \to \infty$.