5.4 Regular singular points

If $P(x_0) = 0$ then x_0 is a <u>singular point</u>.

Then the solution can be undefined for $x = x_0$

Ex) $x^2 y''(x) - 2y(x) = 0$

Which is the singular point?

This is an Euler equation.

Two linear independent solutions are

$$y_1(x) = x^2$$

$$y_2(x) = \frac{1}{x} \quad \text{undefined at } x = 0$$

<u>Regular singular points</u>: (svagt singulära punkter)

$$\lim_{x \to x_0} (x - x_0) \frac{Q(x)}{P(x)} = \lim_{x \to x_0} (x - x_0) p(x) \quad \text{finite}$$

and

$$\lim_{x \to x_0} (x - x_0)^2 \frac{R(x)}{P(x)} = \lim_{x \to x_0} (x - x_0) q(x) \quad \text{finite}$$

That is

$$P(x) = \frac{p_{-1}}{x} + p_0 + p_1 x + p_2 x^2 + \dots$$

$$q(x) = \frac{q_{-2}}{x^2} + \frac{q_{-1}}{x} + q_0 + q_1 x + \dots$$

Do 5.4.2

Do 5.4.20 We see here that the ansatz $\sum_{n=0}^{\infty} a_n x^n = y(x)$ does not work. The Euler equations in 5.5 will give us a hint about the correct ansatz (see 5.6)

5.5 Euler equations

$$L y = x^2 y''(x) + \alpha x y'(x) + \beta y(x) = 0$$

5.5.1 with the method in 5.5.23

$$x^2 y''(x) + 4x y'(x) + 2y(x) = 0$$

$$x > 0$$

Introduce the new variabel

$s = \ln x$. Then

$$\frac{dy}{dx} = \frac{dy}{ds} \cdot \frac{ds}{dx} = \frac{1}{x}\frac{dy}{ds}$$

$$\frac{d^2y}{dx^2} = -\frac{1}{x^2}\frac{dy}{ds} + \frac{1}{x}\frac{d^2y}{ds^2} \cdot \frac{1}{x}$$

Put all this into the equation

$$\frac{x^2}{x^2}\left[y''(s) - y'(s)\right] + \frac{4x}{x}\, y'(s) + 2y(s) = 0$$

$$y''(s) + 3y'(s) + 2y(s) = 0$$

Constant coefficients! That was the reason for the change of variables.

We solve this ODE easily

$$y(s) = Ae^{-s} + Be^{-2s} \quad \text{so}$$

$$y(x) = \frac{A}{x} + \frac{B}{x^2}$$

Since the ODE looks the same for $\xi = -x$ (check!) we can write the solution as

$$y(x) = \frac{A}{|x|} + \frac{B}{|x|^2}$$