5.6) Regular singular point $x_0 = 0$

$$P(x)\, y''(x) + Q(x)\, y'(x) + R(x)\, y(x) = 0$$

$$(*) \quad y''(x) + p(x)\, y'(x) + q(x)\, y(x) = 0$$

Let first $x > 0$.

For a weak singularity we have:

$$\left.\begin{aligned} x\,p(x) &= \sum_{n=0}^{\infty} p_n x^n \\ x^2 q(x) &= \sum_{n=0}^{\infty} q_n x^n \end{aligned}\right\} \text{both analytic!}$$

Multiply $(*)$ with x^2 and approximate $x \cdot p$ and $x^2 \cdot q$ with lowest order terms.

$$x^2 y''(x) + p_0 x\, y'(x) + q_0\, y = 0$$

An Euler equation.

From 5.5 we know that we have to find the roots r_1 and r_2 to the

INDICIAL EQUATION

$$r(r-1) + p_0 r + q_0 = 0$$

Reasonable guess: The two solutions to (*) are

$$(**) \quad \begin{aligned} y_1(x) &= x^{r_1} \cdot \left(\sum_{n=0}^{\infty} a_n x^n \right) \\ y_2(x) &= x^{r_2} \cdot \left(\sum_{n=0}^{\infty} b_n x^n \right) \end{aligned}$$

modifications for larger x

This is often right!
But modifications for $r_1 = r_2$ (logaritmic singularity) and possibly also when $r_1 - r_2$ is an integer.
However one solution is always of this form! Namely the one with largest root r_1.

<u>Frobenius method</u> $\quad y(x) = x^r \sum_{n=0}^{\infty} a_n x^n$

1) Find r_1 and r_2
2) Find the recurrence relation for a_n
3) Which is the radius of convergence?

Why singularities?
End effects due to coordinates Chosen like
$r = 0$
$\cos\theta = \pm 1$

5.6.5 $\quad 3x^2y'' + 2xy' + x^2y = 0$

$$p_0 = 2/3$$
$$q_0 = 0$$

Indicial equation $\quad r(r-1) + \frac{2}{3}r = 0$

$$r_1 = 1/3$$
$$r_2 = 0$$

Both solutions are of the type in (**).

General ansats

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y'(x) = \sum_{n=0}^{\infty} a_n (n+r)\, x^{n+r-1}$$

$$y''(x) = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1)\, x^{n+r-2}$$

All this into the ODE gives

$$3 \sum_{n=0}^{\infty} a_n (n+r)(n+r-1)\, x^{n+r}$$
$$+2 \sum_{n=0}^{\infty} a_n (n+r)\, x^{n+r}$$
$$+ \sum_{n=0}^{\infty} a_n X^{n+r+2} = 0$$

x^r term:

$$x^r \cdot 3 \cdot a_0 \underbrace{\left[\overbrace{r(r-1) + \frac{2}{3} r}^{=F(r), \text{ see } 5.7} + 0 \right]}_{\text{Must be zero! Gives indicial equation}}$$

Recurrence relation:

$$3a_n (n+r)(n+r-1) +$$
$$2a_n (n+r) \quad +$$
$$a_{n-2} \quad = 0 \quad \text{or} \quad 3a_n F(n+r) + a_{n-2} = 0$$

see eq. 8 in 5.7

$$a_n = \frac{-a_{n-2}}{(n+r)(3n+3r-1)}$$

$r=0$: $a_0 = 1$, $a_2 = -\frac{1}{10}$, $a_4 = \frac{1}{440}$

$r = 1/3$ $a_0 = 1$, $a_2 = -1/14$ $a_4 = \frac{1}{728}$

How large is the radius of convergence?

Q! The coefficient in front of x^{r+1} is $a_1 (r+1)(3r+2)$. Gives $r_1 = -1$ och $r_2 = -2/3$. How come? New solutions?

Ex) Solve the equation

$$4x^2 y''(x) - 8x^2 y'(x) + (4x^2+1) y(x) = 0$$

when $x > 0$.

1) Find first $y_1(x) = x^{r_1} \sum_{n=0}^{\infty} a_n x^n$

Try to express it with help of elementary functions.

2) Instead of using a complicated ansats for y_2 (see Th. 5.7.1) put $y_2(x) = y_1(x) \cdot w(x)$ and solve the (simple) ODE for $w(x)$!

Note the classical equations in 5.6.11, 12, 13.

The Legendre equation appears when solving the Schrödinger equation in spherical coordinates

5.7) $$L[y] = x^2 y''(x) + x\left[x p(x)\right] y' + \left[x^2 q(x)\right] y = 0$$

$$x p(x) = p_0 + p_1 x + p_2 x^2 + \dots \qquad x > 0$$

$$x^2 q(x) = q_0 + q_1 x + q_2 x^2 + \dots$$

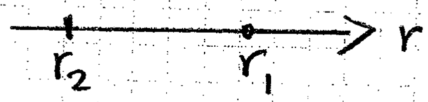
Solution $y(x) = \phi(r, x) = x^r \sum_{n=0}^{\infty} a_n x^n$, $a_0 \neq 0$

$$L[\phi(r,x)] = a_0 F(r) x^r +$$

$$\sum_{n=1}^{\infty} \left\{ F(r+n) a_n + \sum_{k=0}^{n-1} a_k \left[(r+k) p_{n-k} + q_{n-k}\right] \right\} x^{r+n} = 0$$

Since $a_0 \neq 0$! $F(r) = r(r-1) + r p_0 + q_0 = 0$

Indicial equation $F(r) = 0$ has two roots r_1 and r_2. We consider the case when they are real.



$r_1 > r_2$

Three cases to investigate

i) $r_1 \neq r_2$, $r_1 - r_2 \notin \mathbb{N}$. $F(r_2 + n) \neq 0$ when $n > 0$.

$$y_1(x) = x^{r_1}\left(1 + \sum_{n=1}^{\infty} a_n(r_1) x^n\right)$$

$$y_2(x) = x^{r_2}\left(1 + \sum_{n=1}^{\infty} a_n(r_2) x^n\right)$$

ii) $r_1 = r_2$ double root.

$Y_1(x)$ has the same form as above. How is $Y_2(x)$ looking like?

$$L[\phi(r,x)] = a_0 (r-r_1)^2 x^r$$

$$L\left[\frac{\partial \phi}{\partial r}\right](r_1, x) =$$

$$a_0 \left[(r-r_1)^2 x^r \ln x + 2(r-r_1) x^r \right]\Big|_{r=r_1}$$

$= 0$ so $\frac{\partial \phi}{\partial r}$ is the second solution!

$$Y_2(x) = Y_1(x) \ln x + x^{r_1} \sum_{n=1}^{\infty} b_n(r_1) x^n$$

iii) $r_1 > r_2$ $Y_1(x)$ as above.

$r_1 - r_2 \in \mathbb{N}$ Note that $F(r_2 + m) = 0$

$r_1 - r_2 = m$

$$\sum_{k=0}^{m-1} a_k (r_2+k) P_{m-k} + q_{m-k}$$

$= 0$. Then a_m is free. $Y_2(x)$ and Y_1 are of the same type

$\neq 0$ We realise that the ansatz is wrong.

$$Y_2(x) = Y_1(x) \ln x + x^{r_2}\left(1 + \sum_{n=1}^{\infty} c_n(r_2) x^n\right)$$