Mathematics
DFM
Linneaus University

Partial differential equations
4MA103
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Eigenfunctions in a 2D-box

The pictures below show the nodal structure for five different linear combinations of two *degenerate* eigenfunctions to Helmholtz equation for a 2D-box of quadratic shape [1]. The eigenfunctions are forced to be zero at the boundary (Dirichlet condition). Two eigenfunctions are said to be degenerate if they have the same eigenvalue.

If the square has side length π the eigenvalueproblem $(\Delta + k^2)\psi = 0$ has with Dirichlet boundary conditions the following solution

$$k^2 = m^2 + n^2 \tag{1}$$

$$\psi_{mn}(x, y) = A_{mn} sin(mx) sin(ny) \tag{2}$$

Here m and n are positive integers and A_{mn} a constant that is normally fixed by the normalization condition

$$\int\int_D \psi_{mn}^2 dxdy = 1 \tag{3}$$

The domain of integration D is the square $0 \leq x \leq \pi, 0 \leq y \leq \pi$. The eigenfunctions are odd or even with respect to reflections in the lines $x = \frac{\pi}{2}$ and $y = \frac{\pi}{2}$. Quantum physicists speak about eigenfunctions with positive (negative) *parity* if the are even (odd) with respect to such reflections. Can you tell which is the third lowest eigenvalue with (-,-) parity, i.e. odd eigenfunctions both in x and y? Sometimes two different eigenstates have the same k^2-value. As mentioned above this phenomena is called degeneration and is fundamental for the understanding of the stability of atoms and nuclei. In the figure below linear combinations of two degenerate eigenfunctions are taken in the following way

$$\phi(x, y) = cos(\chi)\psi_{mn} + sin(\chi)\psi_{nm} \tag{4}$$

The parameter χ lies between 0 and $\pi/2$. The regions in D where ϕ is negative (positive) is black (white). Can you see which m and n that are used in the figure? Observe that nodal curves, i.e. curves where $\phi = 0$, generally don't cross each other. This observation can be understood in the following way. At a nodal crossings is, besides ϕ, also two partial derivatives of ϕ zero. That means three conditions and that is generally two much for two coordinates to deal with.

References

[1] H. J. Korsch Phys. Lett. A97 (1983) 77.

Out[12]=

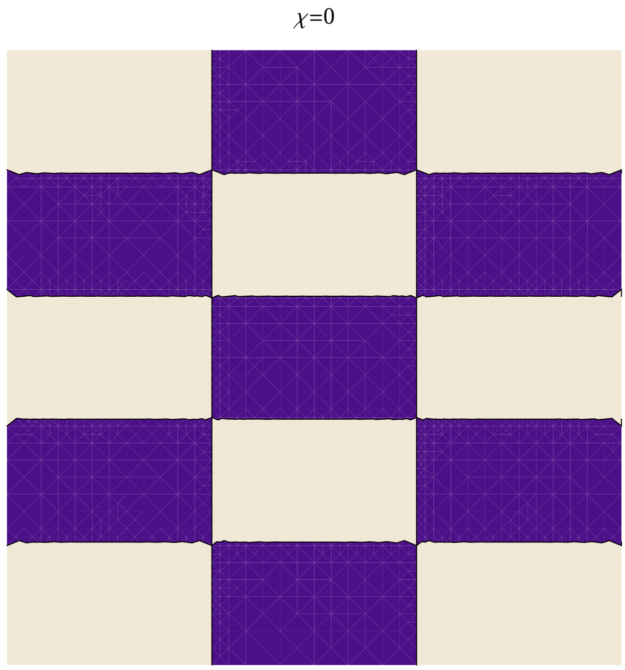
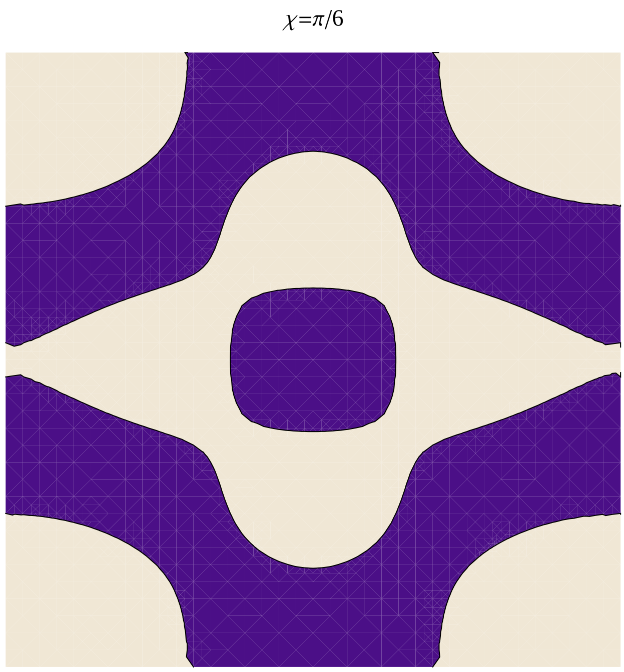
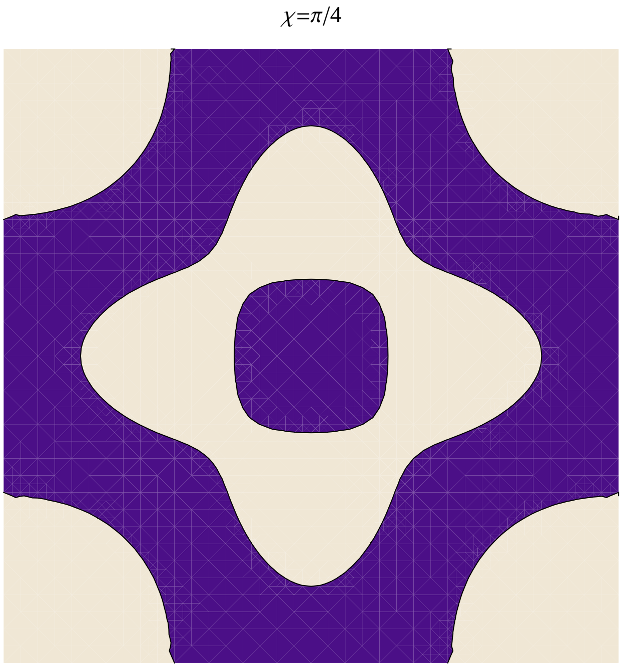
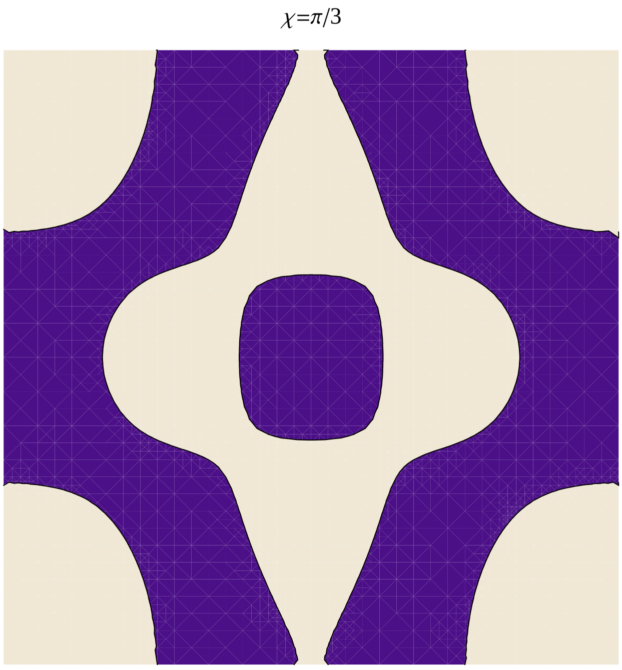


Figure 1: The function $\phi(x,y) = cos(\chi)\psi_{mn} + sin(\chi)\psi_{nm}$ for $\chi = 0$ and for other χ values (see below). In black (white) regions $\phi(x,y)$ is negative (positive). By counting nodal lines for $\chi = 0$ or $\chi = \pi/2$ you can determine m and n. In the most common situation nodal lines dont cross.




$\chi=\pi/4$


$\chi=\pi/3$

$\chi=\pi/2$